



Note

Vizing's conjecture: A two-thirds bound for claw-free graphs



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ABSTRACT

We show that for any claw-free graph G and any graph H , $\gamma(G \square H) \geq \frac{2}{3} \gamma(G) \gamma(H)$, where $\gamma(G)$ is the domination number of G .

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1. Introduction

For basic graph theoretic notation and definitions see Diestel [6]. All graphs $G(V, E)$ are finite, simple, connected, undirected graphs with vertex set V and edge set E . We may refer to the vertex set and edge set of G as $V(G)$ and $E(G)$, respectively.

For any graph $G = (V, E)$, a subset $S \subseteq V$ dominates G if $N[S] = V(G)$. The minimum cardinality of $S \subseteq V$ dominating G is called the *domination number* of G and is denoted $\gamma(G)$. We call a dominating set that realizes the domination number a γ -set.

Definition 1.1. The Cartesian product of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2$ and edge set $E(G_1 \square G_2) = \{(u_1, v_1), (u_2, v_2) : v_1 = v_2 \text{ and } (u_1, u_2) \in E_1, \text{ or } u_1 = u_2 \text{ and } (v_1, v_2) \in E_2\}$.

In 1963, Vadim G. Vizing posed his now famous conjecture [9]: For any pair of graphs G and H ,

$$\gamma(G \square H) \geq \gamma(G) \gamma(H). \quad (1.1)$$

The statement is known for graphs with domination number two [4] and three [8]. Recently, Boštjan Brešar produced a clear and concise new proof of the result for graphs with domination number three [3].

The best current bound for the conjectured inequality was shown in 2010 by Suen and Tarr [7],

$$\gamma(G \square H) \geq \frac{1}{2} \gamma(G) \gamma(H) + \frac{1}{2} \min\{\gamma(G), \gamma(H)\}.$$

In the survey [4], the authors proved a slightly better bound for claw-free graphs, showing that for any claw-free graph G and any graph H , $\gamma(G \square H) \geq \frac{1}{2} \gamma(G)(\gamma(H) + 1)$.

In this note we apply the Contractor–Krop overcount technique [5] to the method of Brešar [3] to show that for any claw-free graph G and any graph H , $\gamma(G \square H) \geq \frac{2}{3} \gamma(G) \gamma(H)$.

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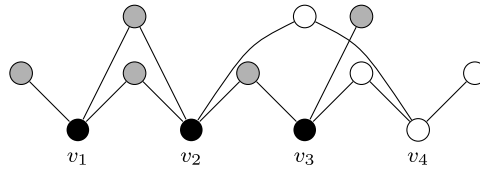


Fig. 1. The Chamber of $Q_{1,2,3}$.

1.1. Notation

A graph G is *claw-free* if G contains no induced $K_{1,3}$ subgraph.

Let $\Gamma = \{v_1, \dots, v_k\}$ be a minimum dominating set of G and for any $i \in [k]$, define the set of *private neighbors* for v_i , $P_i = \{v \in V(G) - \Gamma : N(v) \cap \Gamma = \{v_i\}\}$. For $S \subseteq [k]$, $|S| \geq 2$, we define the *shared neighbors* of $\{v_i : i \in S\}$ as $P_S = \{v \in V(G) - \Gamma : N(v) \cap \Gamma = \{v_i : i \in S\}\}$.

For any $S \subseteq [k]$, say $S = \{i_1, \dots, i_s\}$ where $s \geq 2$, we will usually write P_S as P_{i_1, \dots, i_s} .

For $i \in [k]$, let $Q_i = \{v_i\} \cup P_i$. We call $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ the *cells* of G . For any $I \subseteq [k]$, we write $Q_I = \bigcup_{i \in I} Q_i$ and call $\mathcal{C}(Q_I) = Q_I \cup \bigcup_{S \subseteq I} P_S$ the *chamber* of Q_I . We may write this as \mathcal{C}_I .

In Fig. 1, the black vertices are in the minimum dominating set. The chamber of $Q_{1,2,3}$ is composed of the black and gray vertices.

For a vertex $h \in V(H)$, the G -*fiber* of h , G^h , is the subgraph of $G \square H$ induced by $\{(g, h) : g \in V(G)\}$.

For a minimum dominating set D of $G \square H$, we define $D^h = D \cap G^h$. Likewise, for any set $S \subseteq [k]$, $P_S^h = P_S \times \{h\}$, and for $i \in [k]$, $Q_i^h = Q_i \times \{h\}$. By v_i^h we mean the vertex (v_i, h) . For any $I^h \subseteq [k]$, where I^h represents the indices of some cells in G -fiber G^h , we write \mathcal{C}_{I^h} to mean the chamber of $Q_{I^h}^h$, that is, the set $\bigcup_{i \in I^h} Q_i \cup \bigcup_{S \subseteq I^h} P_S^h$.

For ease of reference, assume that our representation of $G \square H$ is with G on the x -axis and H on the y -axis.

Any vertex $(v, h) = v^h \in G^h$ is *vertically dominated* by D if $(\{v\} \times N_H[h]) \cap D \neq \emptyset$. Vertices that are not vertically dominated are called *vertically undominated*. For $i \in [k]$ and $h \in V(H)$, we say that the cell Q_i^h is *vertically dominated* if $(Q_i \times N_H[h]) \cap D \neq \emptyset$. A cell which is not vertically dominated is *vertically undominated*. Note that all vertices of a vertically undominated cell Q_i^h are dominated by vertices $(u, h) = u^h \in D^h$.

An *independent dominating set* of a graph G is a set of independent (pairwise mutually non-adjacent) vertices which dominate G . The size of a smallest independent dominating set of G is denoted by $i(G)$.

2. Claw-free graphs

We begin with the fundamental result on the domination of claw-free graphs.

Theorem 2.1 (Allan and Laskar [1]). *If G is claw-free, then $i(G) = \gamma(G)$.*

The following fact follows from the definition of claw-free graphs.

Observation 2.2. *For any claw-free graph G with minimum independent dominating set $\{v_1, \dots, v_k\}$, for any $S \subseteq [k]$ with $|S| \geq 3$, $P_S = \emptyset$.*

Our argument, like that of Bartsalkin and German [2], relies on labeling the vertices of a minimum dominating set, D , of $G \square H$ with labels that contain integers from $\{1, \dots, \gamma(G)\}$. Labels may be sets of integers of size one or pairs of distinct integers. We show that every set of labels containing a fixed integer is at least of size $\gamma(H)$. We then control the overcount of vertices by applying the method of Contractor and Krop [5]. This is done by first applying a series of three labelings of the vertices of D . Labels may contain one or two integers and in each successive labeling, we reduce the number of labels with two integers while at the same time maintaining the property that vertices with labels that contain a fixed integer, when projected onto H , form a dominating set of H .

In particular, Labeling 1 gives a singleton label to vertices of D which can be projected onto a fixed dominating set or the private neighbors of the dominating set of G . Other vertices of D are given a paired label. Labeling 2 reduces the number of paired labels that interact with each other in different G -fibers while Labeling 3 reduces the number of paired labels that interact with each other in the same G -fiber.

The resulting relabeled set D satisfies the property that every G -fiber with a certain number of vertices labeled by two integers must contain at least as many vertices labeled by one integer. This allows us to show the claimed lower bound on $|D|$.

Theorem 2.3. *For any claw-free graph G and any graph H ,*

$$\gamma(G \square H) \geq \frac{2}{3} \gamma(G) \gamma(H).$$

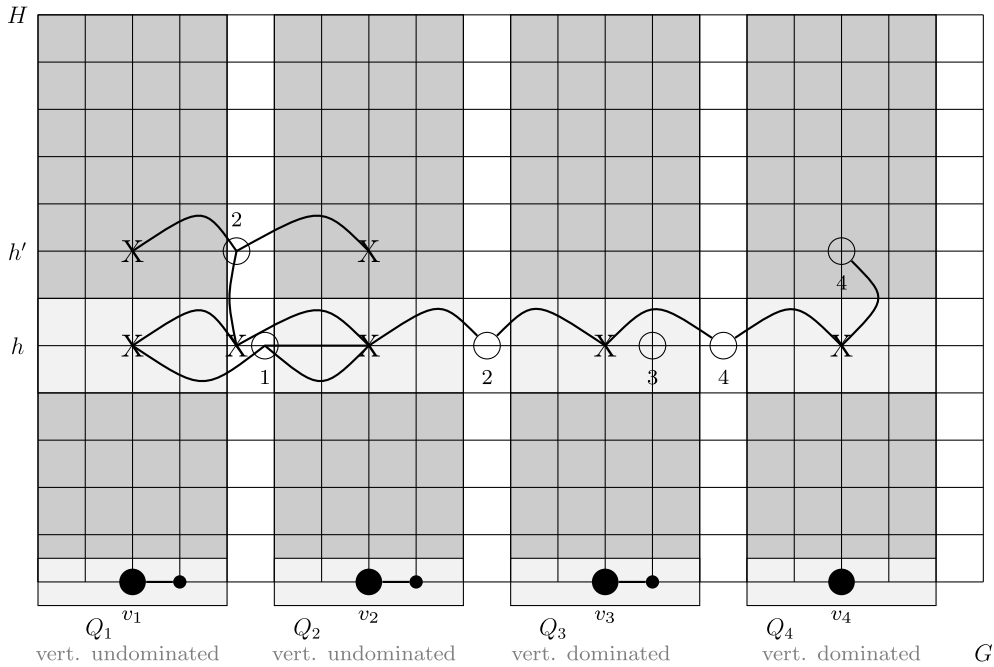


Fig. 2. Labeling 2.

Proof. Let G be a claw-free graph and H any graph. We apply [Theorem 2.1](#) and consider a minimum independent dominating set of G , $\Gamma = \{v_1, \dots, v_k\}$. Let D be a minimum dominating set of $G \square H$.

Our proof is composed of a series of increasingly refining labelings of the vertices of D . In all instances, for any $i, j \in [k]$ and $h \in V(H)$, if $v \in P_{i,j}^h$, then v may be labeled by singleton labels i, j , or paired labels (i, j) .

Our goal is to reduce the number of paired labels as much as possible.

For any $h \in V(H)$, suppose the fiber G^h contains $\ell_h (= \ell)$ vertically undominated cells $U = \{Q_{i_1}^h, \dots, Q_{i_\ell}^h\}$ for some $0 \leq \ell \leq k$. We set $I^h = \{i_1, \dots, i_\ell\}$.

We apply the procedure *Labeling 1* to the vertices of D . If a vertex of D^h for any $h \in H$, is in $Q_{j_1}^h$ for $1 \leq j_1 \leq k$, then we label that vertex by j_1 . If $v \in D^h$ is a shared neighbor of some subset of $\{v_{j_1} : j_1 \in I^h\}$, then by [Observation 2.2](#), it is a member of P_{j_1, j_2}^h for some $j_1, j_2 \in I^h$, and we label v by the pair of labels (j_1, j_2) . If v is a member of $D \cap P_{j_1, j_2}^h$ for $i \in I^h$ and $j_2 \in [k] - I^h$, then we label v by j_1 . If v is a member of $D \cap P_{j_1, j_2}^h$ for $j_1, j_2 \in [k] - I^h$, then we label v by either j_1 or j_2 arbitrarily. This completes *Labeling 1*.

After *Labeling 1*, all vertices of D have a singleton label or a paired label.

We relabel the vertices of D , doing so in D^h for fixed $h \in H$, stepwise, until we exhaust every $h \in H$. This procedure, which we call *Labeling 2*, is described next.

Suppose $v^h \in P_{j_1, j_2}^h \cap D$ for some $j_1, j_2 \in I^h$, $h \in V(H)$, and there exists $y^{h'} \in P_{j_1, j_2}^{h'} \cap D$ for $h' \in V(H)$, $h' \neq h$. The vertex $y^{h'}$ may be labeled by a singleton or paired label, whether *Labeling 2* had been performed on $D^{h'}$ or not.

Suppose that $y^{h'}$ is labeled by a singleton label, say j_1 . Remove the paired label (j_1, j_2) from v^h and relabel v^h by j_2 .

Suppose $y^{h'}$ is labeled by the paired label (j_1, j_2) . Remove the paired label (j_1, j_2) from v^h and then relabel v^h arbitrarily by one of the singleton labels j_1 or j_2 , and then relabel $y^{h'}$ by the other singleton label. This completes *Labeling 2*.

After *Labeling 2*, a vertex v^h of D may have a paired labels (j_1, j_2) if $v^h \in P_{j_1, j_2}^h$ and for any $h' \in N_H(h)$, $D^{h'} \cap P_{j_1, j_2}^{h'} = \emptyset$.

We show an example of some labels after *Labeling 2* in [Fig. 2](#).

Next we describe *Labeling 3*. For every $h \in H$, if D^h contains vertices x and y both with paired labels (j_1, j_2) , for some integers j_1, j_2 , then we relabel x by the label j_1 and y by the label j_2 . For every $h \in H$, if D^h contains vertices x and y with paired label (j_1, j_2) , (j_2, j_3) respectively, for some integers j_1, j_2 , and j_3 , then we relabel y by the label j_3 . If x and y are labeled j_1 and (j_1, j_2) respectively, for some integers j_1, j_2 , we relabel y by j_2 . We apply this relabeling to pairs of vertices of D^h , sequentially, in any order. This completes *Labeling 3*.

For $h \in H$, let S_1^h be the set of vertices of D^h which still have a pair of labels. Notice that after *Labeling 3*, S_1^h is contained in \mathcal{C}_h . For each vertex in S_1^h , we place each component of the paired label on that vertex in the set J_1^h . For example, if S_1^h contains vertices with labels (i_1, i_2) and (i_3, i_4) , then $J_1^h = \{i_1, i_2, i_3, i_4\}$.

Define the index set $I_1^h = [k] - I^h = \{i_{\ell+1}, \dots, i_k\}$ for vertically dominated cells of G^h .

The following observations follow from the definition of claw-free:

- (1) For $j_1, j_2 \in [k] - I^h$, no vertex of $D \cap P_{j_1, j_2}^h$ may dominate any of $v_{j_1}^h, \dots, v_{j_2}^h$. Thus, $\{v_{j_1}^h, \dots, v_{j_2}^h\}$ must be dominated horizontally in G^h by shared neighbors of $\{v_i^h : i \in I^h\}$ from the chamber of $Q_{j_1}^h$.
- (2) If j_1, j_2, j_3, j_4 are distinct elements of $[k]$ and $x \in P_{j_1, j_2}^h, y \in P_{j_3, j_4}^h$, then x is not adjacent to y .
- (3) Similarly, $x \in P_{j_1}^h$ is not adjacent to any $y \in P_{j_2, j_3}^h$.
- (4) By (2), all vertices of $D^h - \mathcal{C}_{j_1}^h$ which are adjacent to some vertex of $\mathcal{C}_{j_1}^h$ must be members of P_i^h for $i \in I_1^h$.
- (5) If a vertex of $\mathcal{C}_{j_1}^h$ is (a) vertically undominated and (b) dominated from outside $\mathcal{C}_{j_1}^h$, then it must be a member of P_j^h for some $j \in J_1^h$, since neither shared neighbors of $\mathcal{C}_{j_1}^h$, nor v_j^h for $j \in J_1^h$, can be adjacent to vertices outside $\mathcal{C}_{j_1}^h$.

Observations (1)–(5) imply the following:

Claim 2.4. *If v is a vertically undominated vertex of $\mathcal{C}_{j_1}^h$ which is not dominated by a shared neighbor (from $\mathcal{C}_{j_1}^h$ or outside $\mathcal{C}_{j_1}^h$), then it is a private neighbor in $\mathcal{C}_{j_1}^h$. Furthermore, v must be dominated by a private neighbor of $\mathcal{C}_{j_1}^h$.*

Set $D_{ij}^h = D^h \cap P_{ij}^h$ for $\ell + 1 \leq j \leq k$ and $D_{i_1}^h = D^h \cap \mathcal{C}_{i_1}^h$. Let $E_{j_1}^h$ be a minimum subset of vertices of $D_{i_1}^h$ so that $(D \cap \mathcal{C}_{j_1}^h) \cup (D \cap N_H(\mathcal{C}_{j_1}^h)) \cup E_{j_1}^h$ dominates $\mathcal{C}_{j_1}^h$. That is, $E_{j_1}^h$ is a minimum set of vertices with neighbors in $\mathcal{C}_{j_1}^h$, which along with the dominating vertices in $\mathcal{C}_{j_1}^h$ and $N_H(\mathcal{C}_{j_1}^h)$, dominate $\mathcal{C}_{j_1}^h$. However, note that due to Labeling 2 and the definition of J_1^h , $D \cap N_H(\mathcal{C}_{j_1}^h)$ is empty. Thus, $E_{j_1}^h$ is a minimum subset of vertices of $D_{i_1}^h$ so that $(D \cap \mathcal{C}_{j_1}^h) \cup E_{j_1}^h$ dominates $\mathcal{C}_{j_1}^h$. By Claim 2.4, $E_{j_1}^h$ is composed of private neighbors of $\mathcal{C}_{j_1}^h$, and hence are labeled by a singleton label.

Claim 2.5. *For every $h \in H$, $|E_{j_1}^h| \geq |S_1^h|$.*

Proof. Suppose not. Set $j = |E_{j_1}^h|$ and $s = |S_1^h|$. Notice that $E_{j_1}^h \cup S_1^h$ dominates $\mathcal{C}_{j_1}^h$. Furthermore, since after Labeling 3 label pairs are disjoint, $|J_1^h| = 2s$. Note that $E_{j_1}^h$ may contain vertices that are shared neighbors of Γ^h , which were relabeled in Labeling 3 to singleton labels. To address this, we define the set of labels of vertices in $E_{j_1}^h$ which had been reduced from paired labels to singleton labels as L^h . If we let $I' = [k] - J_1^h - L^h$, then $E_{j_1}^h \cup S_1^h \cup (\bigcup_{i \in I'} v_i^h)$ dominates G^h . However, such a set contains at most $j + s + k - 2s = j - s + k < k$ vertices, which contradicts the minimality of $\gamma(G)$. \square

By Claim 2.5, D^h contains $|S_1^h|$ vertices labeled by a paired label and at least as many vertices labeled by a singleton label.

Claim 2.6. *For a fixed i , $1 \leq i \leq k$, projecting all vertices such that i is an element of the label (singleton or in a pair) to H produces a dominating set of H .*

Proof. For a fixed $i \in [k]$, if Q_i^k is not vertically dominated, then D^h must contain a vertex adjacent to v_i^h . Such a vertex either contains i in its label or, after Labeling 2, there exists $h' \in V(H)$ adjacent to h , and $u^{h'} \in D^{h'}$ so that $u^{h'}$ has i in its label. In either case, projecting vertices of D with i as an element of their labels onto H produces a dominating set of H . \square

Call the set of such vertices of D labeled i , D_i . Summing over all i we count at least $\gamma(G)\gamma(H)$ vertices of D where we count the members of S_1^h twice and the members of $E_{j_1}^h$ and $D^h - S_1^h - E_{j_1}^h$ once, for every $h \in H$. For a fixed sum $\sum_{i=1}^k |D_i|$, $|D^h|$ is minimized when we maximize the number of dominating vertices that are counted twice. Thus we obtain,

$$\gamma(G)\gamma(H) \leq \sum_{i=1}^k |D_i| \leq 2 \frac{|D|}{2} + \frac{|D|}{2} = \frac{3}{2}|D|. \quad \square$$

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