Peakless Motzkin paths with marked level steps

Naiomi Cameron
ncameron@lclark.edu
Department of Mathematics
Lewis & Clark College
Portland, Oregon USA

Everett Sullivan
everett.n.sullivan@dartmouth.edu
Department of Mathematics
Lewis & Clark College
Portland, Oregon USA

A Motzkin path is any path starting on the x-axis that can make up moves, down moves, and level moves, such that it ends on the x-axis and never goes below the x-axis. We may take a Motzkin path and mark any number of levels steps that lie on the x-axis.

1. Introduction and Background

A *Motzkin path* of length n is a lattice path starting at $(0, 0)$ and ending at $(n, 0)$ that uses up steps $U = (1, 1)$, down steps $D = (1, -1)$, and level steps $L = (1, 0)$, such that the path never goes below the x-axis. The set of Motzkin paths is equivalent to the set of words of length n with letters U, D, and L such that at every index, the number of preceding U's is greater than or equal to the number of preceding D's. Motzkin paths are counted by the Motzkin numbers, which are deeply connected to the Catalan numbers and thus have been the subject of numerous studies over the last forty years. See [**?**] and [**?**] for a couple of the earliest surveys.

In this paper, we wish to draw attention in particular to Motzkin paths that have no peaks, i.e., Motzkin paths that do not contain the subsequence UD. Peakless Motzkin paths are counted by the generalized Catalan numbers, also called the RNA numbers, $1, 1, 1, 2, 4, 8, 17, 37, 82, 185, 423, \ldots$ whose generating function is given by

$$\frac{(1 - x + x^2) - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2}. \tag{1}$$

Peakless Motzkin paths are in bijection with graph theoretic representations of the planar folding of RNA molecules, otherwise known as *RNA secondary structures*. RNA secondary structures and RNA numbers have been the subject of several studies in more recent years, including [**?**] and [**?**].

For the present study, we will consider peakless Motzkin paths where level steps at a certain fixed height are allowed to be distinguished or "marked." For instance, the number of peakless Motzkin paths of length n having k marked level steps on the x-axis is given by the (n, k)th entry of the following infinite lower triangular array:

$$\begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 2 & 3 & 3 & 1 & & & \\ 4 & 6 & 6 & 4 & 1 & & \\ 8 & 13 & 13 & 10 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This array is known to be a *pseudo-involution* in the *Riordan group*. The main objective of this paper is to give a combinatorial interpretation of this fact using Motzkin paths.

The organization of the paper will proceed as follows. The remainder of Section 1 provides the background and definitions needed to explain why the array R above is a pseudo-involution in the Riordan group and to set the stage for the combinatorial interpretation in terms of Motzkin paths. In Section 2, we provide a generating function for the number of Motzkin paths with a prescribed number of level steps occurring on the x-axis and a prescribed number of peaks. In Section 3, we present a combinatorial proof that the array R is a pseudo-involution in the Riordan group by way of an involution on the set of pairs of peakless marked Motzkin paths. In Section 4, we prove an extension of the result from Section 3 to obtain a new class of combinatorial arrays and corresponding combinatorial identities.

1.1 Background on the Riordan group

In order to provide the background motivating this paper, we will briefly describe the Riordan group. For more formative background on the Riordan group, see [**?**].

An element R of the Riordan group is an infinite lower triangular array whose k-th column has generating function $g(x)f^k(x)$, where $k = 0, 1, 2, \ldots$ and $g(x)$, $f(x)$ are generating functions with $g(0) = 1$, $f(0) = 0$. We say R is a *Riordan array* and write $R = (g(x), f(x))$. Multiplication in the Riordan group is matrix multiplication. Since a Riordan group element is determined by a pair of generating functions, the product of Riordan matrices can be described in terms of generating functions as follows

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x))).$$

The identity element of the Riordan group is $(1, x)$ and the inverse of R is given by

$$R^{-1} = (g(x), f(x))^{-1} := \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right) \tag{2}$$

where $\bar{f}$ is the compositional inverse of f. An element R of the Riordan group is a called a **pseudo-involution** if RM has order two, where $M = (1, -x)$.

Proposition 1.1 ([?]). *Let*

$$p(x) = \frac{(1 - x + x^2) - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2}.$$

Then the Riordan array

$$(p(x), xp(x)) = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 3 & 1 \\ 4 & 6 & 6 & 4 & 1 \\ 8 & 13 & 13 & 10 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} =: \langle p_{n,k} \rangle_{n,k\geq 0} \tag{3}$$

is a pseudo-involution. That is,

$$\begin{pmatrix} 1 \\ 1 & -1 \\ 1 & -2 & 1 \\ 2 & -3 & 3 & -1 \\ 4 & -6 & 6 & -4 & 1 \\ 8 & -13 & 13 & -10 & 5 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^2 = \begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or equivalently, for fixed n and l,

$$\sum_{k=0}^{\infty} (-1)^{k+l} p_{n,k}\ p_{k,l} = \begin{cases} 1, & \text{if } n = l \\ 0, & \text{otherwise} \end{cases}$$

Proposition 1.1 was established using Riordan group algebra in [**?**].

We use $p_{n,k}$ to denote the (n, k)-th entry of the Riordan array $(p(x), xp(x))$ from equation (3). In Section 2, we show that $p_{n,k}$ is the number of peakless Motzkin paths of length n having k marked level steps on the x-axis. In Section 3 we provide a combinatorial proof of Proposition 1.1 in terms of peakless marked Motzkin paths.

1.2 Definitions and notation for marked Motzkin paths

As this paper focuses on Motzkin paths with particular features, we wish to make the those features clear in the following definition.

Definition 1.2. Suppose P is a Motzkin path.

- A **flat** in P is a level step that occurs on the x-axis. A flat is preceded by an equal number of up and down steps.
- A **peak** on P is a sequence of an up step followed immediately by a down step.
- A **hill** is a peak that begins on the x-axis, or equivalently is preceded by an equal number of up and down steps.
- We say P is **peakless** if it contains no peaks.
- If s is a step in P, we define the **height** of s to be the difference between the number of up steps and down steps that precede it.
- A **tunnel** in P is a pair of steps (U^*, D^*) where U^* is an up step, D^* is a down step that appears after U^*, both U^* and D^* have the same height and every step between U^* and D^* have a strictly greater height. A step s that occurs between the two steps of a tunnel is said to be *contained* by the tunnel.

Throughout this paper we will allow level steps at a specified height to be distinguished by referring to them as **marked** level steps.

Definition 1.3. A **marked Motzkin path** is a Motzkin path where some level steps may be marked. In relation to words, marked level steps will be represented with the letter M. We say a marked Motzkin path is **base marked** or that it is a **marked base Motzkin path** if marked level steps may only occur at height zero.

We will focus on Motzkin paths that are peakless and based marked.

Definition 1.4. Let $\mathcal{P}_{n,k}^{(h)}$ denote the set of peak-less Motzkin paths of length n with exactly k marked level steps, where marked level steps are only allowed to occur at height h. Then $\mathcal{P}_{n,k}^{(0)}$ denotes the set of peakless base marked Motzkin paths of length n with k marked flats. Moreover, we define $p_{n,k}^{(h)} := |\mathcal{P}_{n,k}^{(h)}|$.

2. GENERATING FUNCTIONS FOR MARKED MOTZKIN PATHS

In this section, we derive the ordinary generating function for $p_{n,k}^{(0)}$, the number of peakless base marked Motzkin paths of length n with k marked flats. We show that this generating function is the same as the generating function for the kth column of the Riordan array in equation (3).

Theorem 2.1. *The coefficient of $x^a m^b \ell^c p^d$ in the ordinary generating function*

$$M(x,m,\ell,p) = \frac{2}{1-2\ell+x^2-px^2+m+\sqrt{(m+px^2-1-x^2)^2-4x^2}},$$

is the number of Motzkin paths with a up steps, a down steps, b level steps not on the x-axis, c flats, and d peaks.

Proof. Let $M(x,m,\ell,p)$ be the generating function of Motzkin paths.

For any non-empty Motzkin path, there is a unique positive integer r such that $(r,0)$ is on the path, and no such smaller positive number is on the path.

Additionally every Motzkin path must start with a up step, which has weight x, or a level step, which has weight ℓ since we start on the x-axis.

In the second case $r=1$, and what follows in another, possibly empty, Motzkin path. Thus the generating function for these Motzkin paths is $\ell M(x,m,\ell,p)$.

In the first case, if the first step is an up step, then the step immediately proceeding the point $(r,0)$ must be a down step. Thus the path must pass through the points $(1,1)$ and $(r-1,1)$. What is between these two points must be another Motzkin path since the same steps must be used and if it goes below 1 it will hit the bottom contradicting the fact that r is the first positive integer r such that $(r,0)$ is a point on the path.

However in this raised Motzkin path there are no flats, and so the weight of flats and all other level steps must be the same. If we want to consider flats and all other level steps to have the same weight, the generating function for these paths is $M(x, m, m, p)$.

Additionally if the raised Motzkin path is empty, then the original path has an up step then a down step and then another Motzkin path, and so has two extra steps and an extra peak.

Thus we break down the case where a Motzkin path begins with an up step into two cases, one where it immediately takes a down step and thus form a peak, and the other where is starts another non-empty Motzkin path.

As demonstrated, the generating function for paths of the first form is

$$x^2 pM(x, m, \ell, p)$$

while the generating function for paths of the second form is

$$x\,(M(x, m, m, p) - 1)\,xM(x, m, \ell, p),$$

which is equal to

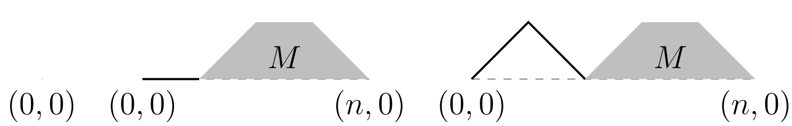
$$x^2 M(x, m, m, p)M(x, m, \ell, p) - x^2 M(x, m, \ell, p).$$

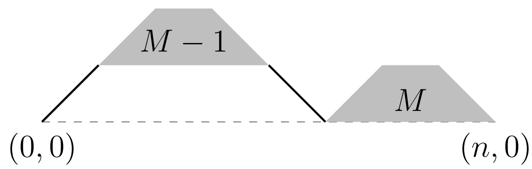
Thus every Motzkin path is either: the empty path given by the generating function 1, a level step followed by another Motzkin path given by the generating function $\ell M(x, m, \ell, p)$, a up step followed by a down step followed by another Motzkin path given by the generating function $x^2 pM(x, m, \ell, p)$, or a up step followed by a raised non-empty Motzkin path followed by a down step and then another Motzkin path given by the generating function $x^2 M(x, m, m, p)M(x, m, \ell, p) - x^2 M(x, m, \ell, p)$.

Since every Motzkin path must be of one of these forms and all the cases are mutually exclusive we have that

$$\begin{aligned} M(x, m, \ell, p) = 1 &+ \ell M(x, m, \ell, p) + x^2 pM(x, m, \ell, p) \\ &+ x^2 M(x, m, m, p)M(x, m, \ell, p) - x^2 M(x, m, \ell, p). \end{aligned}$$

Pictorially, we have that every Motzkin path looks likes exactly one of the following where M is a Motzkin path:





Solving for $M(x,m,\ell,p)$ we get that

$$M(x,m,\ell,p) = \frac{1}{1-\ell-x^2p+x^2-x^2M(x,m,m,p)}.$$

Since

$$M(x,m,m,p) = \left. M(x,m,\ell,p)\right|_{\ell=m},$$

we have that

$$\begin{aligned} M(x,m,m,p) &= 1 + mM(x,m,m,p) + x^2pM(x,m,m,p) \\ &\quad + x^2M(x,m,m,p)^2 - x^2M(x,m,m,p). \end{aligned}$$

Solving for $M(x,m,m,p)$ we get that

$$M(x,m,m,p) = \frac{(1-m-x^2p+x^2) + \sqrt{(m+x^2p-x^2-1)^2-4x^2}}{2x^2}.$$

Substituting back into the formula for $M(x,m,\ell,p)$ and simplifying, we get that

$$M(x,m,\ell,p) = \frac{2}{1-2\ell+x^2-px^2+m+\sqrt{(m+px^2-1-x^2)^2-4x^2}}.$$

□

Corollary 2.2. *The generating function for $p^{(0)}_{n,k}$, the number of peakless Motzkin paths of length n having exactly k marked flats, is $x^kp^{k+1}(x)$ where*

$$p(x) = \frac{(1-x+x^2) - \sqrt{1-2x-x^2-2x^3+x^4}}{2x^2}.$$

Thus, $p_{n,k}$, the (n,k)-th entry of the Riordan array in equation (3), is equal to $p^{(0)}_{n,k}$.

Proof. We will show that the generating function for the sequence $\{|\mathcal{P}^{(0)}_{n,k}|\}_{n=0}^{\infty}$ is

$$\left.\left(\frac{\ell^k}{k!}\frac{\partial^k}{\partial \ell^k}M(x,x,\ell,0)\right)\right|_{\ell=x}$$

Given a peak-less Motzkin path $m \in \mathcal{P}_{n,0}$ it contributes some weight $x^a m^b \ell^c$ to $M(x, m, \ell, 0)$. If we want to count the how many different ways there are to mark k of its flats, then we count the number of ways to pick k of its c flats. Thus the total weight every of peak-less marked Motzkin paths with exactly k marked flats that we can obtain from m is

$$\binom{c}{k} x^a m^b \ell^c p^d = \frac{c(c-1)\cdots(c-k+1)}{k!} x^a m^b \ell^c = \frac{\ell^k}{k!}\frac{\partial^k}{\partial \ell^k}\left(x^a m^b \ell^c\right).$$

Summing these over all peak-less Motzkin paths, give us the multivariate generating function for k-marked flat peak-less Motzkin paths.

Since only the total length of the path matters, m and ℓ is placed with x, which gives the total length of the path. However, since we must take the derivative with respect to ℓ of the function and no other variable, the substitution of x into ℓ must be done afterwards.

Thus the generating function for the sequence $\{|\mathcal{P}^{(0)}_{n,k}|\}_{n=0}^{\infty}$ is

$$\left(\frac{\ell^k}{k!}\frac{\partial^k}{\partial \ell^k} M(x, x, \ell, 0)\right)\Bigg|_{\ell=x}$$

Now, we will show that

$$\left(\frac{\ell^k}{k!}\frac{\partial^k}{\partial \ell^k} M(x, x, \ell, 0)\right)\Bigg|_{\ell=x} = x^k p^{k+1}(x)$$

where

$$p(x) = \frac{(1 - x + x^2) - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2}.$$

First, notice that, by Theorem 2.1, $M(x, x, \ell, 0) = 2(\beta(\ell))^{-1}$ where

$$\beta(\ell) = (1 - 2\ell) + x + x^2 + \sqrt{1 - 2x - x^2 - 2x^3 + x^4}.$$

It is easy to show by induction that

$$\frac{\partial^k}{\partial \ell^k} M(x, x, \ell, 0) = \frac{\partial^k}{\partial \ell^k} 2\beta^{-1}(\ell) = 2^{k+1} k! \beta^{-(k+1)}(\ell)$$

Therefore, we have

$$\begin{aligned}
\left(\frac{\ell^k}{k!}\frac{\partial^k}{\partial \ell^k}M(x,x,\ell,0)\right)\bigg|_{\ell=x} &= \left(\frac{\ell^k}{k!}2^{k+1}k!\beta^{-(k+1)}(\ell)\right)\bigg|_{\ell=x}\\
&= x^k 2^{k+1}\beta^{-(k+1)}(x)\\
&= 2^{k+1}x^k\left(1-x+x^2+\sqrt{1-2x-x^2-2x^3+x^4}\right)^{-(k+1)}\\
&= x^k\left(\frac{2}{1-x+x^2+\sqrt{1-2x-x^2-2x^3+x^4}}\right)^{k+1}\\
&= x^k p^{k+1}(x)
\end{aligned}$$

□

3. AN INVOLUTION ON PAIRS OF PEAKLESS MARKED BASE MOTZKIN PATHS

As in Section 1, we let $p_{n,k}$ denote the (n,k)th entry of the Riordan array (3). We showed in Section 2 that $p_{n,k} = p^{(0)}_{n,k}$, the number of peakless Motzkin paths of length n having exactly k marked flats. We now turn to the first objective of this paper, which is to provide a combinatorial proof of the following identity. Given fixed n, l,

$$\sum_{k=0}^{\infty}(-1)^{k+l}p_{n,k}\ p_{k,l} = \begin{cases}1, & \text{if } n=l\\ 0, & \text{otherwise}\end{cases} \tag{4}$$

This identity is a direct algebraic consequence of Proposition 1.1. However, we wish to prove identity (4) combinatorially, thereby providing means to generalize the identity for a class of arrays in Section 4. To accomplish this task, we will construct an sign-reversing involution $\alpha^*_{n,\ell}$ on the set of pairs of base marked Motzkin paths having k and l marked flats, respectively. Figure 1 summarizes how $\alpha^*_{n,\ell}$ works.

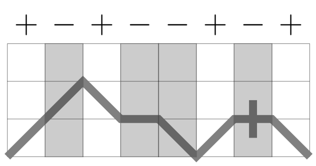
We begin by constructing a map S which creates signed marked Motzkin paths from pairs of base marked Motzkin paths.

Definition 3.1. A **signed** marked Mozkin path is a Mozkin path where each step is designated as being either positive (+) or negative (−).

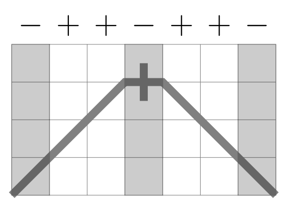
$$\begin{array}{ccc} S(M_1, M_2) & \xrightarrow{\alpha_{n,\ell}} & \alpha_{n,\ell}(S(M_1, M_2)) \\ {\scriptstyle S}\big\uparrow & & \big\downarrow{\scriptstyle S^{-1}} \\ (M_1, M_2) & \xrightarrow{\alpha^*_{n,\ell}} & (N_1, N_2) \end{array}$$

Figure 1: Given $M_1 \in \mathcal{P}^{(0)}_{n,k}$ and $M_2 \in \mathcal{P}^{(0)}_{k,\ell}$, $\alpha^*_{n,\ell}$ is the composition of signed substitution S (as in Definition 3.3), $\alpha_{n,\ell}$, and the unique decomposition from Lemma 3.11.

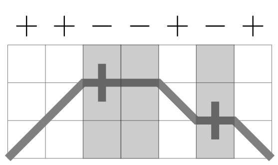
Example 3.2. Here are some examples of signed marked Motzkin paths, where marked levels have a vertical mark in the middle.



$U^+U^-D^+L^-D^-U^+M^-D^+$



$U^-U^+U^+M^-D^+D^+D^-$

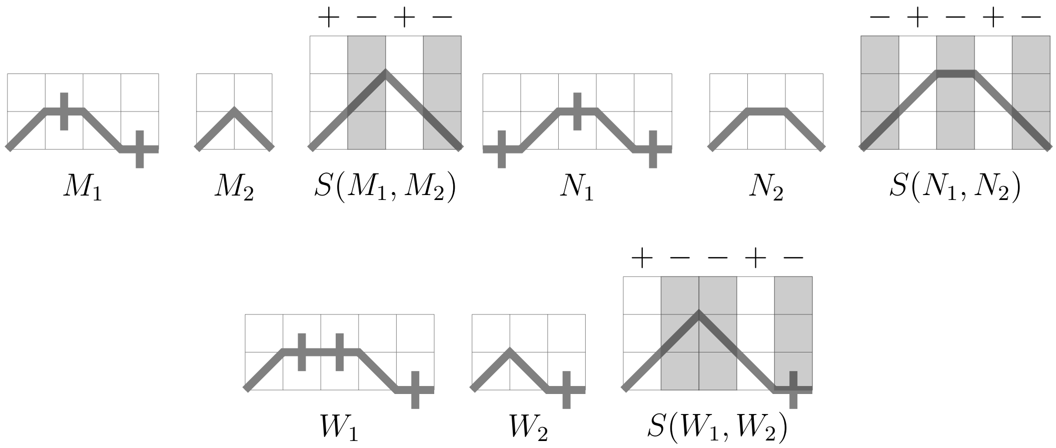


$U^+U^+M^-L^-D^+M^-D^+$

Definition 3.3. Let $\mathcal{M}_{n,k}$ denote the set of marked Motzkin paths of length n with k marked level steps (at any height). Given marked Motzkin paths $M_1 \in \mathcal{M}_{n,k}$ and $M_2 \in \mathcal{M}_{k,\ell}$ we define the **signed substitution** $S(M_1, M_2)$ **of** M_2 **into** M_1 to be the signed marked Motzkin path obtained by first replacing the i^{th} mark level step of M_1 with the i^{th} step of M_2 for $1 \leq i \leq k$, and then designating every position in $S(M_1, M_2)$ where replacement occurred with as negative and all others as positive.

Note that we will also use S as a function from $\mathcal{M}_{n,k} \times \mathcal{M}_{k,\ell}$ to the set of signed paths. Formally this should be written as $S((M, N))$ where $(M, N) \in \mathcal{M}_{n,k} \times \mathcal{M}_{k,\ell}$, however to avoid extra parenthesis we shall write this as $S(M, N)$ as well.

Example 3.4. Here are some examples of signed substitution S, where marked levels have a vertical mark in the middle.



Theorem 3.5. *Given marked Motzkin paths $M \in \mathcal{M}_{n,k}$ and $N \in \mathcal{M}_{k,\ell}$, the sequence of unsigned up and down steps given by $S(M, N)$ is a marked Motzkin path of length n with ℓ marked steps.*

Proof. By substitution, the number of marked steps in $S(M, N)$ will be ℓ. Since all the marked steps in M are replaced by the steps in N, a step will be marked if and only if it is a marked step in M replaced by a marked step in N. Since M has ℓ marked step, $S(M, N)$ will have ℓ marked steps.

Now it remains to show that $S(M, N)$ is a Motzkin path. It suffices to show that at any point along the path that the number of down steps does not exceed the number of up steps, and that the number of downs steps equals the number of up steps.

Let X_i be the i^{th} step of $S(M, N)$, M_i be the i^{th} step of M and N_i be the i^{th} step of N. Let u_i be the number of up steps that occur up to and including X_i and d_i be the number of down steps that occur up to and including X_i Every up or down step in $S(M, N)$ either came from form M_1 and is a positive step, or came from M_2 and is a negative step. Let u_i^+ be the number of positive up steps that occur up to and including X_i and d_i^+ be the number of positive down steps that occur up to and including X_i. Similarly, let u_i^- be the number of negative up steps that occur up to and including X_i and d_i^+ be the number of negative down steps that occur up to and including X_i. Then $u_i^+ + u_i^- = u_i$ and $d_i^+ + d_i^- = d_i$ for all $0 \leq i \leq n$.

Since the up and down steps in M appear as positive up and positive down steps and every positive step came from M, the number of up steps in M up to and including M_i is u_i^+ and the number of down steps in M up to and including M_i is d_i^+. Since M is a Motzkin path the number of down steps does not exceed the number of up steps at any point and the number of up is equal to the number of down steps. Thus $d_i^+ \leq u_i^+$ and $d_n^+ = u_n^+$.

Consider step N_k where $k = u_i^- + d_i^-$. Which is to say that the number of steps up in N up to and including N_k is equal to the number negative step in $S(M, N)$ up to and including X_i.

Since N is a marked Motzkin path the number of down steps does not exceed the number of up steps at any point, and the number of up steps in N is equal to the number of down steps in N. Since the up and down steps in N appear as negative up and negative down steps and every negative step came from N, the number of up steps up to and including N_k is u^- and the number of down steps up to and including N_k is d^-. Thus $d_i^- \leq u_i^-$ and $d_n^- = u_n^-$.

Thus

$$d_i = d_i^+ + d_i^- \leq u_i^+ + u_i^- = u_i \qquad \text{and} \qquad d_n = d_n^+ + d_n^- = u_n^+ + u_n^- = u_n.$$

Thus at every point on the path $S(M, N)$ the number of down steps never exceeds the number of up steps and contains the same number of up and down steps.

Thus the sequence of signed up and down steps given by $S(M, N)$ is a marked Motzkin path of length n with ℓ marked steps. □

Lemma 3.6. *Given a marked base Motzkin path $M \in \mathcal{M}_{n,k}$ and a marked Motzkin path $N \in \mathcal{M}_{k,\ell}$, let the signed substitution of N into M be given by $S(M, N)$. Then for every tunnel (U^*, D^*) in $S(M, N)$, U^* and D^* have the same sign.*

Proof. We proceed by contradiction. Suppose that $S(M, N)$ has a tunnel (U^*, D^*) such that U^* and D^* have different sign. Let X_i be the i^{th} step of $S(M, N)$, M_i be the i^{th} step of M and N_i be the i^{th} step of N. Suppose $X_u = U^*$ and $X_d = D^*$.

We first consider the case where the U^* is a positive and the D^* is a negative. Since X_d is a negative down step, M_d is a marked flat in M, since M is a marked base Motzkin path. Furthermore, M_u is a positive up step. Now consider M_u and M_d, and the tunnel T that contains M_u. The other step in T must occur before or after M_d since it can't be M_d itself.

It can not occur after M_d since that would imply that the flat had height at least one. Thus is must occur before M_d. But if that is the case, then after substitution, the X_u would be paired with the corresponding down step from M and not with X_d.

Thus there can not be a tunnel of a positive up step with a negative down step.

By symmetry, there also can not be a tunnel of a negative up step with a positive down step.

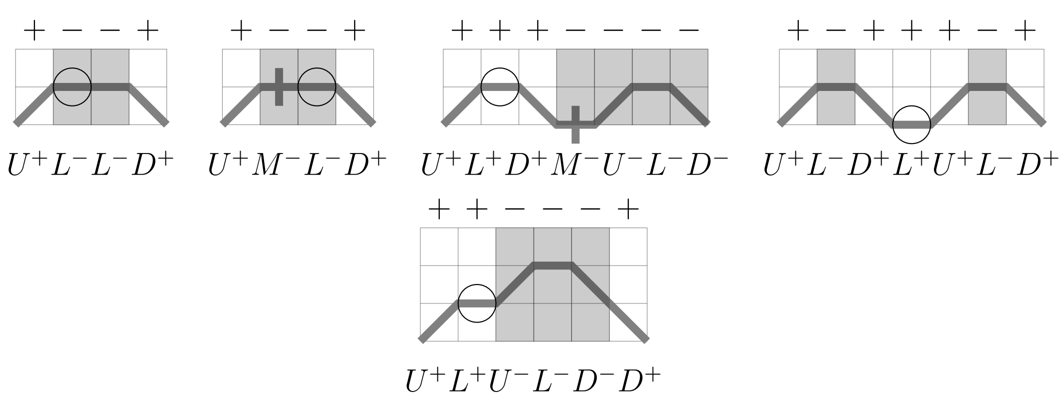
Thus for every tunnel (U^*, D^*) in $S(M, N)$, U^* and D^* have the same sign when M and N is a marked base Motzkin paths. □

We now proceed to construct an involution $\alpha_{n,\ell}$ on the set of signed substitutions of pairs of marked Motzkin paths.

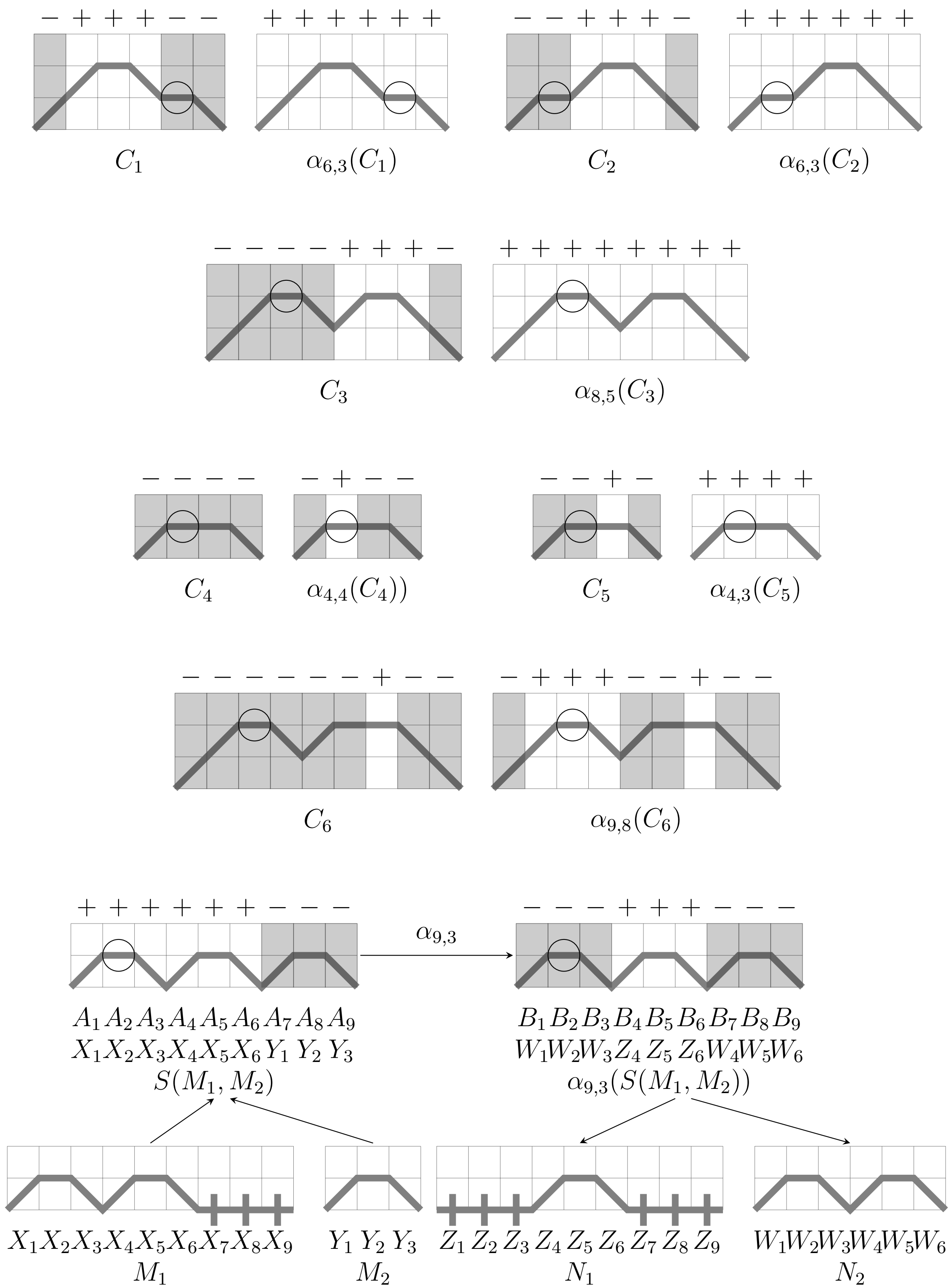
Definition 3.7. Let C be a signed marked Motzkin path. First, we say that a level step L^* of C is the **key level step** if L^* is the first unmarked level step (read left to right) of C such that every unmarked level step that occurs after L^* has equal or greater height. Then we define a function $\alpha_{n,\ell}$ on signed marked Motzkin paths of length n having l marked level steps such that $\alpha_{n,\ell}(C)$ is the path obtained by doing the following:

1. Find the key level step in C (if it exists; if not, $\alpha_{n,\ell}(C) = C$);
2. Switch the sign of the key level step;
3. Switch the sign of both steps in all tunnels (U^*, D^*) of C which contain both the key level step **and** at most one negative level step;
4. Maintain all other steps of C.

Example 3.8. Here are some examples of signed marked Motzkin paths, where marked level steps have a vertical mark in the middle and the key level step is circled.



Example 3.9. Here are examples of $\alpha_{n,\ell}$ applied to signed Motzkin paths.


− + + + − −
+ + + + + +
− − + + + −
+ + + + + +
C_1
$\alpha_{6,3}(C_1)$
C_2
$\alpha_{6,3}(C_2)$
− − − − + + + −
+ + + + + + + +
C_3
$\alpha_{8,5}(C_3)$
− − − −
− + − −
− − + −
+ + + +
C_4
$\alpha_{4,4}(C_4))$
C_5
$\alpha_{4,3}(C_5)$
− − − − − − + − −
− + + + − − + − −
C_6
$\alpha_{9,8}(C_6)$
+ + + + + + − − −
− − − + + + − − −
$\alpha_{9,3}$
$A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9$
$X_1 X_2 X_3 X_4 X_5 X_6 Y_1 Y_2 Y_3$
$S(M_1, M_2)$
$B_1 B_2 B_3 B_4 B_5 B_6 B_7 B_8 B_9$
$W_1 W_2 W_3 Z_4 Z_5 Z_6 W_4 W_5 W_6$
$\alpha_{9,3}(S(M_1, M_2))$
$X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8 X_9$
M_1
$Y_1 Y_2 Y_3$
M_2
$Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9$
N_1
$W_1 W_2 W_3 W_4 W_5 W_6$
N_2

Definition 3.10. Given $n, \ell \geq 0$, let

$$\Psi_{n,\ell} = \bigcup_{k=\ell}^{n} \left(\mathcal{P}_{n,k}^{(0)} \times \mathcal{P}_{k,\ell}^{(0)}\right).$$

and let

$$S(\Psi_{n,\ell}) = \{S(M,N) \mid M \in \mathcal{P}_{n,k}^{(0)}, N \in \mathcal{P}_{k,\ell}^{(0)} \text{ for some } k \text{ where } \ell \leq k \leq n\}$$

denote the image of $\Psi_{n,\ell}$ under the signed substitution function S.

Lemma 3.11. *Given an element $(M_1, M_2) \in \Psi_{n,\ell}$, there exists a unique element $(N_1, N_2) \in \Psi_{n,\ell}$ such that*

$$S(N_1, N_2) = \alpha_{n,\ell}(S(M_1, M_2))$$

Hence, since $S(\Psi_{n,\ell})$ is finite, the function $\alpha_{n,\ell} : S(\Psi_{n,\ell}) \to S(\Psi_{n,\ell})$ is a bijection.

Proof. Let $M_1 \in \mathcal{P}_{n,k}^{(0)}$, $M_2 \in \mathcal{P}_{k,\ell}^{(0)}$ and consider the signed marked Motzkin path $S(M_1, M_2)$. Let X_i be the i^{th} step of M_1, Y_i be the i^{th} step of M_2, A_i be the i^{th} step of $S(M_1, M_2)$, and B_i be the i^{th} step of $\alpha_{n,\ell}(S(M_1, M_2))$. That is,

$$\begin{aligned} M_1 = &\quad X_1 \cdots X_n \\ M_2 = &\quad Y_1 \cdots Y_k \\ S(M_1, M_2) = &\quad A_1 \cdots A_n \\ \alpha_{n,\ell}(S(M_1, M_2)) = &\quad B_1 \cdots B_n \end{aligned}$$

Construction of N_1, N_2: Observe by definition of signed substitution that $S(M_1, M_2)$ has exactly ℓ marked level steps, each of which is negative. Since $\alpha_{n,\ell}$ doesn't add or remove steps and doesn't change the sign of marked level steps, we have that $\alpha_{n,\ell}(S(M_1, M_2))$ must have length n and ℓ marked level steps all of which are negative.

Define $N_1 := Z_1 \cdots Z_n$, where Z_i is an unsigned marked level step if B_i is a negative step, and an unsigned step of the same type as B_i if B_i is a positive step.

In order to define N_2, first let k^* denote the number of negative steps in $\alpha_{n,\ell}(S(M_1, M_2))$. Then consider the set $\{j_1, \cdots, j_{k^*}\}$ of indices of the negative steps of $\alpha_{n,\ell}(S(M_1, M_2))$, where $j_1 < \cdots < j_{k^*}$. We can now define $N_2 := W_1 \cdots W_{k^*}$ where W_i is an unsigned step of the same type as B_{j_i}.

By construction, we see that N_1 has length n with k^* marked level steps, and N_2 has length k^* with ℓ marked level steps. Furthermore, by observation of $\alpha_{n,\ell}(S(M_1, M_2))$,

N_1 and N_2 are the only paths that could be combined via signed substitution to form $\alpha_{n,\ell}(S(M_1, M_2))$. It suffices to show that $N_1 \in \mathcal{P}^{(0)}_{n,k^*}$ and $N_2 \in \mathcal{P}^{(0)}_{k^*,\ell}$.

N_1 and N_2 are marked Motzkin paths: To show that N_1 and N_2 are marked Motzkin paths it suffices to show that at any point along the path that the number of down steps does not exceed the number of up steps, and that the number of downs steps equals the number of up steps.

By Theorem 3.5 we know that $S(M_1, M_2)$ is a marked Motzkin path. Since $\alpha_{n,\ell}$ only changes the signs of steps, we know that $\alpha_{n,\ell}(S(M_1, M_2))$ is also a signed marked Motzkin path. By theorem 3.6 we have that for every tunnel (U^*, D^*) in $S(M_1, M_2)$, the two steps have the same sign. Observe that if $\alpha_{n,\ell}$ changes the sign of one of the steps in the tunnel it must change the other. Thus for every tunnel (U^*, D^*) in $\alpha_{n,\ell}(S(M_1, M_2))$, the two steps have the same sign.

For each $i = 1, \ldots, n$, let u_i^+, u_i^-, d_i^+, and d_i^- be the number of positive up steps, negative up step, positive down steps, and negative up step that occur up to and including B_i in $\alpha_{n,\ell}(S(M_1, M_2))$ respectively. Consider all down steps that occur up to and including B_i. Consider all the tunnels that contain these down steps. Each of these tunnels contain a up step of the same sign as the down step. Thus for every down step up to and including B_i there is a corresponding up step of the same sign which is also before B_i, and so we have that $d_i^+ \leq u_i^+$ and $d_i^- \leq u_i^-$. When $i = n$ we have that $d_n^+ = u_n^+$ and $d_n^- = u_n^-$ since every down step in the entire path has been accounted for and every down step is in a tunnel with an up step.

By construction, the number of up steps and down steps preceding Z_i in N_1 is u_i^+ and d_i^+ respectively, so N_1 is a marked Motzkin path. Similarly, by construction, the number of up steps and down steps preceding W_i in N_2 is $u_{j_i}^-$ and $d_{j_i}^-$ respectively, so N_2 is a marked Motzkin path.

N_1 and N_2 are based marked: We will show that every marked level step in N_1 and N_2 has height zero.

First, we consider N_2. If W_i is a marked level step in N_2, then B_{j_i} is a negative marked level step in $\alpha_{n,\ell}(S(M_1, M_2))$ Since $\alpha_{n,\ell}$ only changes the signs of steps and doesn't change the sign of marked level steps, A_{j_i} must be a negative marked level step in $S(M_1, M_2)$. If A_{j_i} is a negative marked level step, it must be the result of substituting a marked level step of M_1 with a marked level step of M_2.

Since M_2 and M_1 are both based marked, the number of up and down steps that come before the marked level step in M_2 must be equal. Thus the number of negative up and negative down steps that come before A_{j_i} in $S(M_1, M_2)$ are equal. Since the marked level step in M_2 replaces a marked flat in M_1 the number of positive up and positive down steps that come before A_{j_i} in $S(M_1, M_2)$ are equal. Thus A_{j_i} has height zero in $S(M_1, M_2)$. Thus B_{j_i} has height zero in $\alpha_{n,\ell}(S(M_1, M_2))$. Since the number of negative up and negative down steps that come before B_{j_i} in $\alpha_{n,\ell}(S(M_1, M_2))$ are equal, W_i has height zero in N_2. Thus N_2 is based marked.

Now we consider N_1. We want to show that every marked level step in N_1 has height zero. If Z_i is a marked level step in N_1, then B_i is a negative step in $\alpha_{n,\ell}(S(M_1, M_2))$ and A_i is either a positive step or negative step in $S(M_1, M_2)$. So we have two cases to consider:

1. A_i is a negative step. Then X_i is a marked level step in M_1. Since M_1 is based marked, X_i has an equal number of up and down steps before it. This means A_i has an equal number of positive up and positive down steps before it. Since $\alpha_{n,\ell}$ changes the signs of up and down steps in pairs B_i has an equal number of positive up and positive down steps before it. Thus Z_i is preceded by an equal number of up and down steps, meaning Z_i has height zero.

2. A_i is a positive step. Suppose towards a contradiction that X_i does not have height zero. Then there are more up steps then down steps before X_i. Thus there are more positive up steps before B_i than positive down steps. Thus there is a tunnel (B_u, B_v) that contains B_i such that B_u and B_v are positive steps.

   Thus there is a tunnel (A_u, A_v) that contains A_i. Since A_i is a positive step and B_i is a negative step A_i must be an up or down step. Furthermore, the key level step must be a positive step, since $\alpha_{n,\ell}$ switch a step from positive to negative. Thus A_u and A_v must also be positive steps since otherwise we would simultaneously have $\alpha_{n,\ell}$ turn a negative step into a positive step. Without loss of generality suppose that A_i is a down step. Thus there is a tunnel (A_w, A_i) where A_w is a positive step. We observe that (A_u, A_v) must contain A_w as well. Since $\alpha_{n,\ell}$ changed the sign of step A_i it must be that (A_w, A_i) contains the key level step. Thus (A_u, A_v) contains the key level step, which is a positive step. Since A_u and A_v are positive, it must be that B_u and B_v are negative steps, a contradiction. Thus X_i has height zero. Thus X_i is a marked flat. Thus N_1 is based marked.

N_1 and N_2 are peakless: Suppose towards a contradiction that N_1 had a down step occur immediately after an up step. Then there are steps Z_i and Z_{i+1} such that Z_i is an up step and Z_{i+1} is a down step. Thus B_i and B_{i+1} are a positive up and positive down step respectively. Thus A_i and A_{i+1} are a positive up and positive down step respectively since they can't contain the key level step. Thus X_i and X_{i+1} are an up and down step respectively. But then M_1 has a peak, a contradiction. Thus N_1 is peakless.

Suppose towards a contradiction that N_2 had a down step occur immediately after an up step. Then there are steps W_i and W_{i+1} such that W_i is an up step and W_{i+1} is a down step. Thus B_{j_i} and $B_{j_{i+1}}$ are a negative up and negative down step respectively. Observe that every step B_u where $j_i < u < j_{i+1}$ must be a positive step and therefore $(B_{j_i}, B_{j_{i+1}})$ is a tunnel.

Thus $(A_{j_i}, A_{j_{i+1}})$ is a tunnel. Now either $(A_{j_i}, A_{j_{i+1}})$ contains the key level step in $S(M_1, M_2)$ or it doesn't. So we have two cases to consider:

1. $(A_{j_i}, A_{j_{i+1}})$ does not contain the key level step. Then the signs of A_{j_i} and $A_{j_{i+1}}$ would not change, nor would any step that they contain. Thus A_{j_i} and $A_{j_{i+1}}$ are negative steps with no negative steps between them. Thus there would be steps Y_u and Y_{u+1} that are an up and down step respectively. Thus M_2 would have a peak, a contradiction.

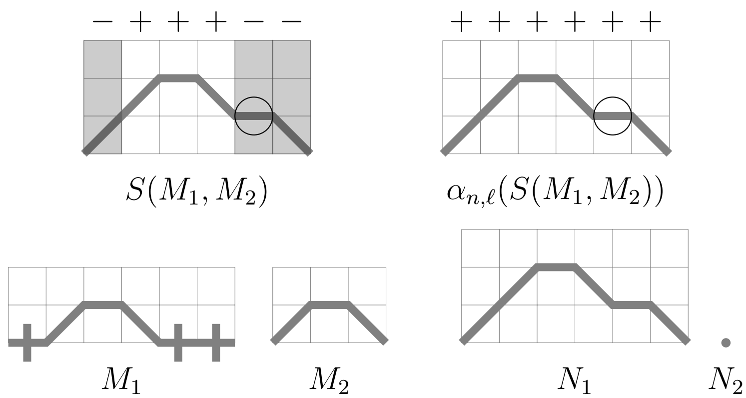
2. $(A_{j_i}, A_{j_{i+1}})$ contains the key level step A_{i^*}. Suppose the key level step was a positive step. Then B_{i^*} is a negative step such that $j_i < i^* < j_{i+1}$, which contradicts that every step between B_{j_i} and $B_{j_{i+1}}$ is a positive step. Thus the key level step is a negative step. Observe that A_{j_i} and $A_{j_{i+1}}$ must be negative step since otherwise they would be positive steps with a negative key step which would mean that B_{j_i} and $B_{j_{i+1}}$ would be positive steps.

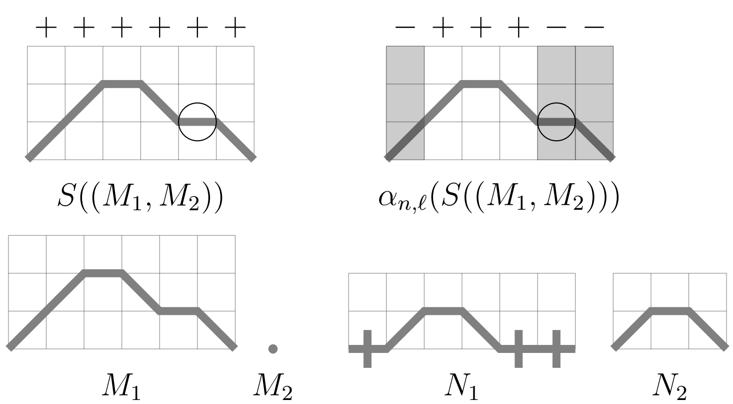
   Furthermore, $(A_{j_i}, A_{j_{i+1}})$ must contain a negative step $A_{i^{**}}$ other the key level step since otherwise by $\alpha_{n,\ell}$, B_{j_i} and $B_{j_{i+1}}$ would be positive steps. But then $B_{i^{**}}$ is a negative step contain in $(B_{j_i}, B_{j_{i+1}})$ a contradiction.

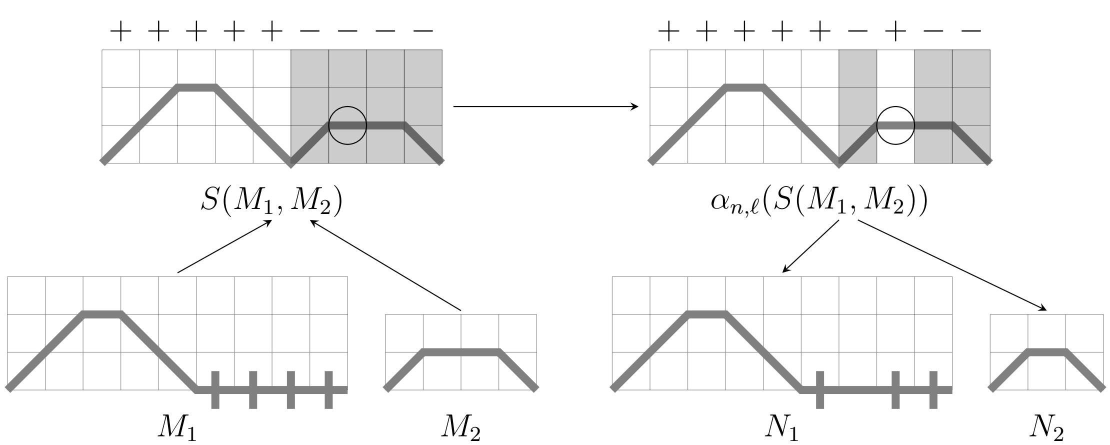
Thus the key level step can be neither positive nor negative, which is impossible. Therefore, N_2 must be peakless.

We have concluded that N_1 and N_2 are peakless base marked Motzkin paths with $N_1 \in \mathcal{P}^{(0)}_{n,k^*}$ and $N_2 \in \mathcal{P}^{(0)}_{k^*,\ell}$. As a consequence we see that $\mathrm{Im}(\alpha_{n,\ell}|_{S(\Psi_{n,\ell})}) \subseteq S(\Psi_{n,\ell})$. □

Example 3.12. Here are a number of examples of signed substitutions of Motzkin paths being put through $\alpha_{n,\ell}$ and being decomposed:


− + + + − −
+ + + + + +
$S(M_1, M_2)$
$\alpha_{n,\ell}(S(M_1, M_2))$
M_1
M_2
N_1
N_2


+ + + + + +
− + + + − −
$S((M_1, M_2))$
$\alpha_{n,\ell}(S((M_1, M_2)))$
M_1
M_2
N_1
N_2


+ + + + + − − − −
+ + + + + − + − −
$S(M_1, M_2)$
$\alpha_{n,\ell}(S(M_1, M_2))$
M_1
M_2
N_1
N_2

Lemma 3.13. *For all n, ℓ, the function $\alpha_{n,\ell}$ is an involution on $S(\Psi_{n,\ell})$.*

Proof. This is a direct corollary of lemma 4.4. □

Finally, to achieve the objective of this section, we will associate a "sign" v with each element of $\Psi_{n,\ell}$ and show that $\alpha^*_{n,\ell}$ is a sign-reversing involution on $\Psi_{n,\ell}$. In fact, we will show that $\alpha^*_{n,\ell}$ is a sign-reversing involution on

$$\Psi^{(h)}_{n,\ell} := \bigcup_{k=\ell}^{n} \left(\mathcal{P}^{(0)}_{n,k} \times \mathcal{P}^{(h)}_{k,\ell}\right).$$

Definition 3.14. Given $\ell \le k \le n$ and a pair $(M, N) \in \mathcal{P}^{(0)}_{n,k} \times \mathcal{P}^{(h)}_{k,\ell}$ we define the function v such that

$$v\,(M, N) = (-1)^{k+\ell}$$

Thus v is -1 if the total number of marked level steps in the pair (M, N) is odd, and 1 otherwise.

Theorem 3.15. *Let $\alpha^*_{n,\ell}$ be the function on $\Psi_{n,\ell}$ such that $\alpha^*_{n,\ell}((M_1, M_2)) = (N_1, N_2)$ from Lemma 3.11. Then $\alpha^*_{n,\ell}$ is an involution of $\Psi_{n,\ell}$ such that for any $(M_1, M_2) \in \Psi_{n,\ell}$ either $\alpha^*_{n,\ell}(M_1, M_2) = (M_1, M_2)$ or $v(\alpha^*_{n,\ell}(M_1, M_2)) = -v(M_1, M_2)$.*

Proof. This is a direct corollary of 4.5. □

Lemma 3.16. *Let F be the set of fixed points of $\alpha^*_{n,\ell} : \Psi^{(h)}_{n,\ell} \to \Psi^{(h)}_{n,\ell}$. That is,*

$$F = \{(M, N) \in \Psi^{(h)}_{n,\ell} \mid \alpha^*_{n,\ell}(M, N) = (M, N)\}$$

. Then

$$\sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M, N) = \sum_{(M,N)\in F} v(M, N)$$

Proof. This is a direct corollary of 4.6. □

Lemma 3.17. *Given $(M_1, M_2) \in \Psi_{n,\ell}$ we have that $\alpha^*_{n,\ell}(M_1, M_2) = (M_1, M_2)$ if and only if $M_1 = M_2$ is the peakless marked base Motzkin path consisting of exactly n marked flats in a row.*

Proof. Suppose that M_1 and M_2 are the same peakless marked base Motzkin paths of exactly n marked flats in a row. Then $S(M_1, M_2)$ is a path of n marked flats all signed with negative. Thus $S(M_1, M_2)$ has no unmarked level step, and thus no key level step. Thus $\alpha_{n,\ell}(S(M_1, M_2)) = S(M_1, M_2)$. Since the decomposition is unique $\alpha^*_{n,\ell}(M_1, M_2) = (M_1, M_2)$.

Now conversely, suppose that $\alpha^*_{n,\ell}(M_1, M_2) = (M_1, M_2)$. Then it must be that $\alpha_{n,\ell}(S(M_1, M_2)) = S(M_1, M_2)$. This can only happen if there is no key level steps since otherwise that step would switch sign and there would be a different pair. Thus M_1 and M_2 have no unmarked level steps.

Additionally M_1 and M_2 have no up or down steps. Suppose they did. Then they must have a tunnel. But every tunnel of a based marked peakless path must contain an unmarked level step. If the tunnel didn't, then it would either contain a marked level step which violates the based marked condition, or it would contain only up and down steps which would imply the existence of a peak witch violates the peakless condition. Thus M_1 and M_2 can't have any tunnels. And so M_1 and M_2 have no up or down steps.

Thus M_1 and M_2 have only marked level steps. Since M_1 has n steps, it therefore must have n marked flats, and so $M_1 \in \mathcal{P}^{(0)}_{n,n}$. In ordered for M_2 to be substituted into M_1 it must have length equal to the number of marked steps in M_1, thus $M_2 \in \mathcal{P}^{(0)}_{n,\ell}$. Since M_2 has only marked level steps we also have that $\ell = n$. And so $M_1, M_2 \in \mathcal{P}^{(0)}_{n,n}$ and contain only marked level steps. □

Theorem 3.18. *Let n, ℓ be any non-negative integers and $p_{n,k}$ be the number of peakless Motzkin paths of length n having exactly k marked level steps occurring on the x-axis. Then*

$$\sum_{k=0}^{\infty} (-1)^{k+\ell} p_{n,k}\ p_{k,\ell} = \begin{cases} 1 & \text{if } n = \ell \\ 0 & \text{if } n \neq \ell \end{cases}$$

Proof. First, observe that since $p_{n,k} = 0$ if $k > n$ and $p_{k,\ell} = 0$ if $\ell > k$, we know that $p_{n,k}p_{k,\ell} = 0$ unless $\ell \leq k \leq n$. Hence,

$$\sum_{k=0}^{\infty} (-1)^{k+\ell} p_{n,k}\ p_{k,\ell} = \sum_{k=\ell}^{n} (-1)^{k+\ell} p_{n,k}\ p_{k,\ell} = \sum_{k=\ell}^{n} \sum_{(M,N)\in\mathcal{P}^{(0)}_{n,k}\times\mathcal{P}^{(0)}_{k,\ell}} v(M, N) \tag{5}$$

$$= \sum_{(M,N)\in\Psi_{n,\ell}} v(M, N) \tag{6}$$

From Lemma 3.16 with $h = 0$, we know that

$$\sum_{(M,N)\in\Psi_{n,\ell}} v(M,N) = \sum_{(M,N)\in F} v(M,N)$$

From Lemma 3.17 we know that F is non-empty if and only if $n = \ell$, so

$$\sum_{(M,N)\in\Psi_{n,\ell}} v(M,N) = \begin{cases} \displaystyle\sum_{(M,N)\in F} v(M,N), & \text{if } n = \ell \\ 0, & \text{if } n \neq \ell \end{cases}$$

If $n = \ell$, then F contains exactly one element, i.e., the pair (M,N) where both M and N consist of $n = \ell$ marked flats. In this case, $v(M,N) = (-1)^{2n} = 1$. Hence, with these observations, equation (6) leads us to the result:

$$\sum_{k=0}^{\infty} (-1)^{k+\ell} p_{n,k}\ p_{k,\ell} = \begin{cases} 1 & \text{if } n = \ell \\ 0 & \text{if } n \neq \ell \end{cases}$$

□

4. AN INVOLUTION ON PAIRS OF PEAKLESS MOTZKIN PATHS MARKED AT FIXED HEIGHT

In the section, we will extend the result in Theorem 3.18 by considering pairs of paths (M,N) where $M \in \mathcal{P}^{(0)}_{n,k}$ and $N \in \mathcal{P}^{(h)}_{k,\ell}$ for any non-negative integer h.

Definition 4.1. Given $n, \ell \geq 0$, let

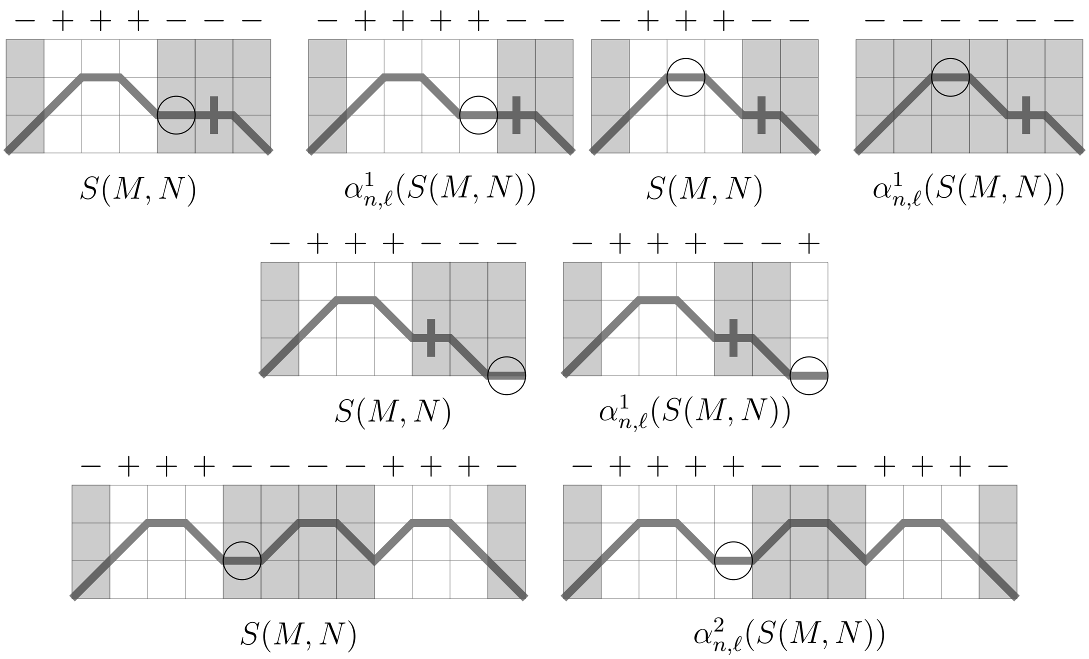
$$\Psi^{(h)}_{n,\ell} = \bigcup_{k=\ell}^{n} \left(\mathcal{P}^{(0)}_{n,k} \times \mathcal{P}^{(h)}_{k,\ell}\right).$$

and let

$$S\left(\Psi^{(h)}_{n,\ell}\right) = \{S\left(M,N\right) \mid M \in \mathcal{P}^{(0)}_{n,k}, N \in \mathcal{P}^{(h)}_{k,\ell} \text{ for some } k \text{ where } \ell \leq k \leq n\}$$

That is, $S\left(\Psi^{(h)}_{n,\ell}\right)$ is the set of all signed substitutions that result from substituting a peakless marked Motzkin path of length k and ℓ marked level steps, where every marked level step has height h, into a peakless marked base Motzkin path of length n with k marked flats.

Example 4.2. Here are a few examples of $\alpha_{n,\ell}^{(h)}$ acting on $S(M,N)$ where $M \in \mathcal{P}_{n,k}^{(0)}$ and $N \in \mathcal{P}_{k,\ell}^{(h)}$ for some fixed h.



Lemma 4.3. *Given* $(M_1, M_2) \in \Psi_{n,\ell}^{(h)}$ *there exists a unique element* $(N_1, N_2) \in \Psi_{n,\ell}^{(h)}$ *such that*

$$S(N_1, N_2) = \alpha_{n,\ell}^{(h)}(S(M_1, M_2))$$

Proof. Consider the signed marked Motzkin path $S(M_1, M_2)$. Let X_i be the i^{th} step of M_1, Y_i be the i^{th} step of M_2, A_i be the i^{th} step of $S(M_1, M_2)$, and B_i be the i^{th} step of $\alpha_{n,\ell}^{(h)}(S(M_1, M_2))$.

Observe by definition of signed substitution that every marked level step in $S(M_1, M_2)$ must be a negative step. Since $\alpha_{n,\ell}^{(h)}$ doesn't add or remove steps and doesn't change the sign of level steps, we have that $\alpha_{n,\ell}^{(h)}(S(M_1, M_2))$ must have length n and ℓ marked level steps all of which are negative.

Then we define N_1 to be the sequence of steps $Z_1 \cdots Z_n$, where Z_i is an unsigned marked level step if B_i is a negative step, and an unsigned step of the same type as B_i if B_i is a positive step.

In order to define N_2 we first let $\overline{k}$ denote the number of negative steps in $\alpha_{n,\ell}^{(h)}(S(M_1, M_2)$. We may then consider the set $\{j_1, \cdots, j_{\overline{k}}\}$ to be the set of indices of the negative steps of $\alpha^{(h)}_{n,\ell}(S(M_1, M_2)$ where $j_1 < \cdots < j_{\overline{k}}$. We can now define N_2 to be the sequence of steps $W_1 \cdots W_{\overline{k}}$ where W_i is an unsigned step of the same type as B_{j_i}.

By the observations of $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$ we have that N_1 has length n with $\overline{k}$ marked level steps, and N_2 has length $\overline{k}$ with ℓ marked level steps.

Observe that by construction N_1 and N_2 are the only paths that could be combined via signed substitution to form $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$. It suffices to show that $N_1 \in \mathcal{P}^{(0)}_{n,k}$ and $N_2 \in \mathcal{P}^{(h)}_{k,\ell}$.

To show that N_1 and N_2 are marked Motzkin paths it suffices to show that at any point along the path that the number of down steps does not exceed the number of up steps, and that the number of downs steps equals the number of up steps.

By theorem 3.5 we know that $S(M_1, M_2)$ is a marked Motzkin path. Since $\alpha^{(h)}_{n,\ell}$ only changes the signs of steps, we know that $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$ is also a signed marked Motzkin path. By theorem 3.6 we have that for every tunnel (U^*, D^*) in $S(M_1, M_2)$, the two steps have the same sign. Observe that if $\alpha^{(h)}_{n,\ell}$ changes the sign of one of the steps in the tunnel it must change the other. Thus for every tunnel (U^*, D^*) in $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$, the two steps have the same sign.

Let u_i^+, u_i^-, d_i^+, and d_i^- be the number of positive up steps, negative up step, positive down steps, and negative up step that occur up to and including B_i in $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$ respectively.

Consider step B_i. Consider all down steps that occur up to and including B_i. Consider all the tunnels that contain these down steps. Each of these tunnels contain a up step of the same sign as the down step. Thus for every down step up to and including B_i there is a corresponding up step of the same sign which is also before B_i, and so we have that $d_i^+ \leq u_i^+$ and $d_i^- \leq u_i^-$. When $i = n$ we have that $d_n^+ = u_n^+$ and $d_n^- = u_n^-$ since every down step in the entire path has been accounted for and every down step is in a tunnel with an up step.

By construction the number of up steps and down steps up to and before Z_i in N_1 is u_i^+ and d_i^+ respectively. Thus N_1 is a marked Motzkin path.

By construction the number of up steps and down steps in up to and before W_i in N_2 is $u_{j_i}^-$ and $d_{j_i}^-$ respectively. Thus N_2 is a marked Motzkin path.

We now show that N_1 is based marked and N_2 has marked sideways steps only at height h.

Consider a marked level step W_i in N_2. Then B_{j_i} is a negative marked level step in

$\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$

Consider step A_{j_i}, since $\alpha^{(h)}_{n,\ell}$ only changes the sign of a step, S_{j_i} must also be a marked level step. Furthermore A_{j_i} must be a negative step since $\alpha^{h}_{n,j}$ doesn't change the sign of marked level steps. Thus A_{j_i} is a negative marked level step in $S(M_1, M_2)$. Since A_{j_i} is a negative marked level step, it must be the result of substituting a marked level step from M_2 into a marked level step from M_1.

Since $M_1 \in \mathcal{P}^{(0)}_{n,k}$ and $M_2 \in \mathcal{P}^{(h)}_{k,\ell}$, the difference of up and down steps that come before the marked level step in M_2 must be h. Thus the difference of negative up and negative down steps that come before A_{j_i} in $S(M_1, M_2)$ is h. Since the marked level step in M_2 is substituted into a marked flat in M_1 the number of positive up and positive down steps that come before A_{j_i} in $S(M_1, M_2)$ are equal. Thus A_{j_i} has height h in $S(M_1, M_2)$. Thus B_{j_i} has height h in $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$. Since the difference of negative up and negative down steps that come before B_{j_i} in $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$ is h, W_i has height h in N_2 Thus the marked level step in N_2 is at height h.

Thus $N_2 \in \mathcal{P}^{(h)}_{k,\ell}$.

Consider a marked level step Z_i in N_1. Then B_i a negative step in $\alpha^{(h)}_{n,\ell}(S(M_1, M_2))$.

Now A_i is either a positive step or negative step in $S(M_1, M_2)$. We break the problem in cases:

Case 1: A_i is a negative step. Then X_i is a marked level step in M_1. Since M_1 is based marked, X_i has an equal number of up and down steps before it. Thus A_i has an equal number of positive up and positive down steps before it. Thus B_i has an equal number of positive up and positive down steps before it since $\alpha^{(h)}_{n,\ell}$ changes the signs of up and down steps in pairs. Thus Z_i has an equal number of up and down steps before it. Thus Z_i is a marked flat.

Case 2: A_i is a positive step. Suppose towards a contradiction that X_i does not have height zero. Then there are more up steps then down steps before X_i. Thus there are more positive up steps before B_i than positive down steps. Thus there is a tunnel (B_u, B_v) that contains B_i such that B_u and B_v are positive steps.

Thus there is a tunnel (A_u, A_v) that contains A_i. Since A_i is a positive step and B_i is a negative step A_i must be an up or down step. Furthermore, the key level step must be a positive step, since $\alpha^{(h)}_{n,\ell}$ switch a step from positive to negative. Thus A_u and A_v must also be positive steps since otherwise we would simultaneously have $\alpha^{(h)}_{n,\ell}$ turn a negative step into a positive step. Without loss of generality suppose that A_i is a down step. Thus there is a tunnel (A_w, A_i) where A_w is a positive step. We observe that (A_u, A_v) must contain A_w as well. Since $\alpha_{n,\ell}^{(h)}$ changed the sign of step A_i it must be that (A_w, A_i) contains the key level step. Thus (A_u, A_v) contains the key level step, which is a positive step. Since A_u and A_v are positive, it must be that B_u and B_v are negative steps, a contradiction.

Thus X_i has height zero. Thus X_i is a marked flat. Thus N_2 is based marked.

We now show that N_1 and N_2 are peakless.

Suppose towards a contradiction that N_1 had a down step occur immediately after an up step. Then there are steps Z_i and Z_{i+1} such that Z_i is an up step and Z_{i+1} is a down step. Thus B_i and B_{i+1} are a positive up and positive down step respectively. Thus A_i and A_{i+1} are a positive up and positive down step respectively since they can't contain the key level step. Thus X_i and X_{i+1} are an up and down step respectively. But then M_1 has a peak, a contradiction. Thus N_1 is peakless.

Suppose towards a contradiction that N_2 had a down step occur immediately after an up step. Then there are steps W_i and W_{i+1} such that W_i is an up step and W_{i+1} is a down step. Thus B_{j_i} and $B_{j_{i+1}}$ are a negative up and negative down step respectively. Observe that every step B_u where $j_i < u < j_{i+1}$ must be a positive step and therefore $(B_{j_i}, B_{j_{i+1}})$ is a tunnel.

Thus $(A_{j_i}, A_{j_{i+1}})$ is a tunnel. Now either $(A_{j_i}, A_{j_{i+1}})$ contains the key level step in $S(M_1, M_2)$ or it doesn't. We break the problem into cases:

Case 1: $(A_{j_i}, A_{j_{i+1}})$ does not contain the key level step. Then the signs of A_{j_i} and $A_{j_{i+1}}$ would not change, nor would any step that they contain. Thus A_{j_i} and $A_{j_{i+1}}$ are negative steps with no negative steps between them. Thus there would be steps Y_u and Y_{u+1} that are an up and down step respectively. Thus M_2 would have a peak, a contradiction.

Case 2: $(A_{j_i}, A_{j_{i+1}})$ contains the key level step A_{i^*}. Suppose the key level step was a positive step. Then B_{i^*} is a negative step such that $j_i < i^* < j_{i+1}$, which contradicts that every step between B_{j_i} and $B_{j_{i+1}}$ is a positive step. Thus the key level step is a negative step. Observe that A_{j_i} and $A_{j_{i+1}}$ must be negative step since otherwise they would be positive steps with a negative key step which would mean that B_{j_i} and $B_{j_{i+1}}$ would be positive steps. We also have that $(A_{j_i}, A_{j_{i+1}})$ can not contain a marked step since otherwise the marked step would be a negative step, and so there would be a corresponding negative marked step that was contained by $(B_{j_i}, B_{j_{i+1}})$.

Furthermore, $(A_{j_i}, A_{j_{i+1}})$ must contain a negative step $A_{i^{**}}$ other the key level step since otherwise by $\alpha_{n,\ell}^{(h)}$, B_{j_i} and $B_{j_{i+1}}$ would be positive steps. But then $B_{i^{**}}$ is a negative step contain in $(B_{j_i}, B_{j_{i+1}})$ a contradiction. Thus the key level step can be neither positive nor negative, which is impossible.

Thus N_2 is peakless.

Thus N_1 and N_2 are peakless base marked Motzkin paths. As a consequence we see that for all $\mathrm{Im}(\alpha_{n,\ell}^{(h)}) \subseteq S\left(\Psi_{n,\ell}^{(h)}\right)$. □

Lemma 4.4. *For all $n \geq \ell$, $h \geq 0$ the function $\alpha_{n,\ell}^{(h)}$ is an involution on $S\left(\Psi_{n,\ell}^{(h)}\right)$.*

Proof. Given an arbitrary element $S(M, N) \in S\left(\Psi_{n,\ell}^{(h)}\right)$ there are three cases to consider: the key level step doesn't exist, the key level step exists and is a positive step, and the key level step exists and is a negative step.

Case 1: Suppose the key level step doesn't exist. Then $\alpha_{n,\ell}^{(h)}(S(M, N)) = S(M, N)$, and so

$$\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M, N))) = S(M, N).$$

Case 2: Suppose the key level step exists and is a positive step. It suffices to show that every step has the same sign when $\alpha_{n,\ell}^{(h)}$ is applied twice. Observe that every step is exactly one of the following types: the key level step, level steps that are not the key level step, steps in tunnels that don't contain the key level step or contain a marked level step, and steps in tunnels that contain the key level step and no marked level steps.

Consider the key level step. Then it switches sign every time $\alpha_{n,\ell}^{(h)}$ is applied, and so it switches signs twice. Thus the key level step has the same sign in both $S(M, N)$, and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M, N)))$.

Consider level steps that are not the key level step. Since $\alpha_{n,\ell}^{(h)}$ only changes the sign of the key level steps and tunnels, these are unaffected by $\alpha_{n,\ell}^{(h)}$. Thus they have the same sign in $S(M, N)$ and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M, N)))$.

Consider steps in tunnels that don't contain the key level step or contain a marked level step. Recall that $\alpha_{n,\ell}^{(h)}$ can only change the sign of steps in a tunnel if the tunnel contains key step and no marked level steps. Since these tunnels don't they are unaffected by $\alpha_{n,\ell}^{(h)}$. Thus they have the same sign in $S(M, N)$ and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M, N)))$.

Consider steps in tunnels that do contain the key level step and no marked level steps. Suppose that both steps in a given tunnel are positive steps. Observe that the tunnel contains no negative steps since that would contradict the fact that M is based marked. Since the key level step is a positive step and we have a tunnel of positive steps that contain it, the key level step and the steps in the tunnel will becomes negative steps after applying $\alpha_{n,\ell}$. Now the tunnel has negative steps and contains exactly one negative step which is the key level steps. Thus the steps in the tunnel will become positive steps after applying $\alpha_{n,\ell}^{(h)}$. Thus tunnels with positive steps that contain the key level step have the same sign in $S(M,N)$ and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M,N)))$.

Suppose that both steps in a given tunnel are negative steps. Observe that the tunnel contains a negative sideways step since otherwise N would have a peak. Since the key level step and steps in the tunnel are of opposite sign the steps in the tunnel are not changed by applying $\alpha_{n,\ell}^{(h)}$.

After applying $\alpha_{n,\ell}^{(h)}$ the key level step is a negative step, but the tunnel now contains at least two negative steps, and so the sign of the steps in the tunnel won't change after applying $\alpha_{n,\ell}^{(h)}$ again. Thus tunnels with negative steps that contain the key level step have the same sign in $S(M,N)$ and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M,N)))$.

Thus every step in $S(M,N)$ will remain the same when $\alpha_{n,\ell}^{(h)}$ is applied twice. Thus

$$\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M,N))) = S(M,N).$$

Case 3: Suppose the key level step exists and is a negative step. It suffices to show that every step has the same sign when $\alpha_{n,\ell}^{(h)}$ is applied twice. Observe that every step is exactly one of the following types: the key level step, level steps that are not the key level step, steps in tunnels that don't contain the key level step or contain a marked level step, and steps in tunnels that contain the key level step and no marked level steps.

Consider the key level step. Then it switches sign every time $\alpha_{n,\ell}^{(h)}$ is applied, and so it switches signs twice. Thus the key level step has the same sign in both E, and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M,N)))$.

Consider level steps that are not the key level step. Since $\alpha_{n,\ell}^{(h)}$ only changes the sign of the key level steps and tunnels, these are unaffected by $\alpha_{n,\ell}^{(h)}$. Thus they have the same sign in $S(M,N)$ and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M,N)))$.

Consider steps in tunnels that don't contain the key level step or contain a marked level step. Recall that $\alpha_{n,\ell}^{(h)}$ can only change the sign of steps in a tunnel if the tunnel contains key step and contain no marked level steps. Since these tunnels don't they are unaffected by $\alpha_{n,\ell}^{(h)}$. Thus they have the same sign in $S(M,N)$ and $\alpha_{n,\ell}^{(h)}(\alpha_{n,\ell}^{(h)}(S(M,N)))$.

Consider steps in tunnels that do contain the key level step and no marked level steps. Observe that any tunnel which contains the key level step can't have positive steps since the key level step is a negative step and a negative step contained in a tunnel with positive steps would mean that M is not based marked.

Thus both steps in a given tunnel are negative steps. Suppose that a given tunnel contains a negative step that is not the key level step. Then the tunnel contains at least two negative steps and so is unaffected by applying $\alpha^{(h)}_{n,\ell}$. Applying $\alpha^{(h)}_{n,\ell}$ again will leave the tunnel unaffected again since the key level step and the tunnel will be of opposite sign.

If the given tunnel contains no other negative steps other than the key level step, applying $\alpha^{(h)}_{n,\ell}$ will change the sign of the steps in the tunnel. But then the tunnel will contain positive steps and contain the key level step which is also a positive step. Thus applying $\alpha^{(h)}_{n,\ell}$ again will change the sign of the steps in the tunnel back to negative.

Thus every step in $S(M,N)$ will remain the same when $\alpha^{(h)}_{n,\ell}$ is applied twice. Thus

$$\alpha^{(h)}_{n,\ell}(\alpha^{(h)}_{n,\ell}(S(M,N))) = S(M,N).$$

Thus $\alpha^{(h)}_{n,\ell}$ is an involution of $S\left(\Psi^{(h)}_{n,\ell}\right)$. □

Theorem 4.5. *Let $\alpha^*_{n,\ell}$ be the function on $\Psi^{(h)}_{n,\ell}$ such that $\alpha^*_{n,\ell}((M_1,M_2)) = (N_1,N_2)$ from Lemma 4.3. Then $\alpha^*_{n,\ell}$ is an involution of $\Psi^{(h)}_{n,\ell}$ such that for any $(M_1,M_2) \in \Psi^{(h)}_{n,\ell}$ either $\alpha^*_{n,\ell}(M_1,M_2) = (M_1,M_2)$ or $v(\alpha^*_{n,\ell}(M_1,M_2)) = -v(M_1,M_2)$.*

Proof. Suppose $(M_1,M_2) \in \mathcal{P}^{(0)}_{n,k} \times \mathcal{P}^{(h)}_{k,\ell}$ for some k. Consider $S(M_1,M_2)$ and $\alpha_{n,\ell}(S(M_1,M_2))$. Now $\alpha^*_{n,\ell}(M_1,M_2) = (N_1,N_2)$ where

$$S(N_1,N_2) = \alpha_{n,\ell}(S(M_1,M_2)).$$

Now, by definition $\alpha^*_{n,\ell}(N_1,N_2) = (L_1,L_2)$ where

$$S(L_1,L_2) = \alpha_{n,\ell}(S(N_1,N_2)) = \alpha_{n,\ell}(\alpha_{n,\ell}(S(M_1,M_2)))$$

By Lemma 4.4 we have that

$$S(L_1,L_2) = \alpha_{n,\ell}(\alpha_{n,\ell}(S(M_1,M_2))) = S(M_1,M_2).$$

By Lemma 4.3 the decomposition is unique and so

$$(L_1,L_2) = (M_1,M_2).$$

Thus

$$\alpha^*_{n,\ell}(\alpha^*_{n,\ell}(M_1, M_2)) = \alpha^*_{n,\ell}(N_1, N_2) = (L_1, L_2) = (M_1, M_2).$$

Thus $\alpha^*_{n,\ell}$ is an involution of $\Psi^{(h)}_{n,\ell}$.

Now let $M_1 \in \mathcal{P}^{(0)}_{n,k}$, $M_2 \in \mathcal{P}^{(h)}_{k,\ell}$ be such that $(M_1, M_2) \neq \alpha^*_{n,\ell}(M_1, M_2)$.

By definition of $\alpha_{n,\ell}$ we know that if $S(M_1, M_2)$ has exactly k negative steps then $\alpha_{n,\ell}(S(M_1, M_2))$ has $k \pm b$ negative steps where b is odd (since an odd number of steps had their sign switched). Thus $N_1 \in \mathcal{P}^{(0)}_{n,k\pm b}$, and $N_2 \in \mathcal{P}^{(h)}_{k\pm b,\ell}$ and

$$v(\alpha^*_{n,\ell}(M_1, M_2)) = v(N_1, N_2) = (-1)^{k\pm b+\ell} = (-1)^{k+\ell}(-1)^{\pm b} = -v(M_1, M_2).$$

That is, if $(M_1, M_2) \neq \alpha^*_{n,\ell}(M_1, M_2)$, then

$$v(\alpha^*_{n,\ell}(M_1, M_2)) = -v(M_1, M_2).$$

□

Lemma 4.6. *Let F be the set of fixed points of $\alpha^*_{n,\ell} : \Psi^{(h)}_{n,\ell} \to \Psi^{(h)}_{n,\ell}$. That is,*

$$F = \{(M, N) \in \Psi^{(h)}_{n,\ell} \mid \alpha^*_{n,\ell}(M, N) = (M, N)\}$$

. Then

$$\sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M, N) = \sum_{(M,N)\in F} v(M, N)$$

Proof. From Theroem 4.5 we know that $\alpha^*_{n,\ell}$ is an v-reversing involution on $\Psi^{(h)}_{n,\ell}$ and thus a bijection on $\Psi^{(h)}_{n,\ell}$. Thus

$$2 \sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M, N) = \sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M, N) + v(\alpha^*_{n,\ell}(M, N)).$$

Let $F \subseteq \Psi^{(h)}_{n,\ell}$ be the set of fixed points of $\alpha^*_{n,\ell}$. Then

$$\begin{aligned}\sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M, N) + v(\alpha^*_{n,\ell}(M, N)) &= \sum_{(M,N)\in F} v(M, N) + v(\alpha^*_{n,\ell}(M, N)) \\ &+ \sum_{(M,N)\in\Psi^{(h)}_{n,\ell}\setminus F} v(M, N) + v(\alpha^*_{n,\ell}(M, N)) \\ &= 2 \sum_{(M,N)\in F} v(M, N),\end{aligned}$$

since by Theorem 4.5, $v(M,N) + v(\alpha^*_{n,\ell}(M,N)) = 0$, for any element $(M,N) \in \Psi^{(h)}_{n,\ell} \setminus F$. Hence,

$$\sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M,N) = \sum_{(M,N)\in F} v(M,N)$$

□

Theorem 4.7. *Let n,ℓ be any non-negative integers and $p^{(h)}_{n,k}$ be the number of peakless Motzkin paths of length n having exactly k marked level steps, where only level steps at height h are allowed to be marked. Then the ordinary generating function over n for*

$$\sum_{k=0}^{\infty} (-1)^{k+\ell} p^{(0)}_{n,k}\ p^{(h)}_{k,l}$$

is

$$\begin{cases} 1 & \text{if } \ell = 0 \\ x^{2h+\ell}\left(\sum_{i=0}^{h} x^{2i}\right)^{\ell-1} & \text{otherwise} \end{cases}$$

Proof. First, observe that since $p^{(h)}_{n,k} = 0$ if $k > n$ and $p^{(h)}_{k,\ell} = 0$ if $\ell > k$, we know that $p^{(0)}_{n,k}\ p^{(h)}_{k,\ell} = 0$ unless $\ell \le k \le n$. Hence,

$$\sum_{k=0}^{\infty} (-1)^{k+\ell} p^{(0)}_{n,k}\ p^{(h)}_{k,\ell} = \sum_{k=\ell}^{n} (-1)^{k+\ell} p^{(0)}_{n,k}\ p^{(h)}_{k,l} = \sum_{k=\ell}^{n} \sum_{(M,N)\in\mathcal{P}^{(0)}_{n,k}\times\mathcal{P}^{(h)}_{k,\ell}} v(M,N) \tag{7}$$

$$= \sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M,N) \tag{8}$$

By Theorem 3.15 we have that $\alpha_{n,\ell}$ is a v-reversing involution and from Lemma 3.16, we know that

$$\sum_{(M,N)\in\Psi^{(h)}_{n,\ell}} v(M,N) = \sum_{(M,N)\in F} v(M,N)$$

where

$$F = \{(M,N) \in \Psi^{(h)}_{n,\ell} \mid \alpha^*_{n,\ell}(M,N) = (M,N)\}.$$

By definition of $\alpha^*_{n,\ell}$ we know that in order for (M,N) to be in F, neither M nor N can have unmarked level steps.

Suppose that $(M, N) \in F$ and $\ell = 0$. Then N has no marked level steps. Then N can have no up or down steps since then it would either have a peak or a sideways step. Thus N must be the empty path. Thus M has no marked level steps and so it is the empty path as well. Thus the only fixed point with no marked steps in N is the pair of empty Motzkin Paths. Hence, if $\ell = 0$, the ordinary generating function for $\sum_{(M,N)\in F} v(M, N)$ over n is 1.

Now suppose $\ell \geq 1$. Since M is based marked and cannot have any level steps it must be a path consisting of $n = k$ marked level steps. We also have that N has at least one marked level step. Since marked steps must occur at height h in N the first h steps of N must be up steps (otherwise there would be an unmarked level step or a peak) and similarly the last h steps of N must be down steps. Thus N must be of the form

$$N = \underbrace{U \cdots U}_{h \text{ steps}} \bar{L} P_{j_1} \bar{L} P_{j_2} \cdots \bar{L} P_{j_{\ell-1}} \bar{L} \underbrace{D \cdots D}_{h \text{ steps}}$$

where there are ℓ marked level steps $\bar{L}$ and each P_{j_i} is some sequence of steps that do not contain any marked level steps. Thus P_{j_i} must only contain up and down steps. If P_{j_i} has up steps, they cannot appear first since that would imply a peak or unmarked level step. Thus each P_{j_i} must be of the form

$$P_{j_i} = \underbrace{D \cdots D}_{j_i \text{ steps}} \underbrace{U \cdots U}_{j_i \text{ steps}}$$

where $0 \leq j_i \leq h$ for each $i = 1, \ldots, \ell - 1$.

Since M is a sequence of marked steps, its length must be equal to N's length. Hence, for $\ell \geq 1$, $(M, N) \in F$ if and only if

$$(M, N) = (\underbrace{\bar{L} \cdots \bar{L}}_{n \text{ steps}}, \underbrace{U \cdots U}_{h \text{ steps}} \bar{L} P_{j_1} \bar{L} P_{j_2} \cdots \bar{L} P_{j_{\ell-1}} \bar{L} \underbrace{D \cdots D}_{h \text{ steps}})$$

where $0 \leq j_i \leq h$, $P_{j_i} = \underbrace{D \cdots D}_{j_i \text{ steps}} \underbrace{U \cdots U}_{j_i \text{ steps}}$, for $i = 1, \ldots, \ell - 1$, and

$$n = 2h + \ell + 2\sum_{i=1}^{\ell-1} j_i.$$

In this case, then $v(M, N) = (-1)^{n+\ell} = 1$ and

$$\sum_{k=0}^{\infty} (-1)^{k+\ell} p_{n,k}^{(0)} \, p_{k,\ell}^{(h)} = \sum_{(M,N)\in F} v(M, N) = |F|$$

whose ordinary generating function over n is

$$x^{2h+\ell}\left(\sum_{i=0}^{h} x^{2i}\right)^{\ell-1}.$$

□